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NON-EUCLIDEAN GEOMETRIES AND ALGORITHMS OF LIVING BODIES†

S. V. PETUKHOV

Department of Biomechanics, Mechanical Engineering Research Institute, U.S.S.R. Academy of Sciences, Griboedov Str. 4, Moscow 101830, U.S.S.R.

Abstract—The importance is shown of non-Euclidean geometrical symmetric transformations and iterative algorithms for the structuring of supramolecular biological bodies. The variety of kinematics of biological movements is related with the "cyclomeric polymorphicity", or restructuring of the iterative algorithm in the biological structure. The author believes that the morphogenetic significance of iterative algorithms in biology is attributable to the mechanisms of interaction in biological layers of tissues and replication of supramolecular structures. The geometrical fundamentals of classical biomorphology need expansion and a generalized biomorphology has to be developed by replacing the conventional similarity symmetries by broader ranges of higher-order transformations (notably Möbius and projective). The progress of theoretical biology is today contingent on more extensive use of group-theoretic methods incorporating higher-order symmetries.

1. THE ERLANGER PROGRAM AND GEOMETRIZATION IN BIOLOGY

One hundred and fifteen years ago F. Klein, who then was 23 years old, put forward the famous Erlanger program which established an essentially new, group invariant view of geometry. Since then geometry has been the science of invariant groups of transformations and the Euclidean geometry, one of the possible geometries, each using its own set of transformations. These are similarity transformations in the Euclidean geometry, affine transformations in the affine geometry, Möbius, or circular, transformations in the conformal geometry and projective transformations in the projective geometry [e.g. 1, 2].

This approach restored to a significant degree the unity of geometry lost by the mid-nineteenth century because of the development of its new fields, clarified the interrelationships of different geometries and made it possible to construct new ones.

The revolution in geometry brought about by the Erlanger program was later extended to the physics and to philosophical views of space. The very fundamentals of human perception of space changed and ever since have been related to the concept of a mathematical group of transformations. H. Poincaré formulated this relationship in very simple terms, space is a group. The advent of the special theory of relativity gave birth to a new term, "geometrization of physics" which stood for the fact that formally this theory is a theory of invariants of some group of transformations (Poincaré–Lorentz group), or a geometry. The ideas of geometrization of physics, representation and description of its theories in the language of invariants of groups of transformations were extended to quantum mechanics, the theory of conservation laws, the theory of elementary particles and other physical fields. The group invariant approach and symmetry concepts become a cornerstone of today's group theoretical thinking [1].

This revolution entailed a revision of the philosophical fundamentals of physics; the view of what the initial laws of physics should be has changed. Since Newton the laws of nature had been formulated as differential equations; this changed completely. For basic physics the major initial laws are laws of symmetry. About this change Yu. B. Rumer and A. I. Fet, Soviet theoreticians, wrote

> "The development of physics in recent years has reversed, in a sense, the relation between equations of motion and symmetry groups. Now the symmetry group of a physical system is in the forefront, the concepts of this group and its subgroup contain the most fundamental information on the system. Consequently, groups become a primary, most profound element in a physical description of nature." [1, p. 4]

†The research reported in this article was partially presented in a motion picture *Cyclic Groups of Transformations in Biomechanics* (filmed by Soyuzvuzfilm with the author's script).

Also, in the words of H. Weyl [3], "the symmetry method is the guiding principle of today's mathematics and its applications".

"Nothing is more fruitful than cognition of oneself", wrote R. Descartes [4]. But what does cognition imply? To know basic biological phenomena in terms of mathematical natural sciences is to interpret these phenomena and their laws in a language of more profound concepts which are characteristic of mathematical natural sciences. The mainstream for biology to bridge the gap to exact sciences is penetration of group invariant concepts and methods into biology through study of biological symmetries. The characteristic headlines of papers on mathematical and theoretical biology are "The concept of a group and the perception theory" [5], "Biological similarity and the group theory" [6], "Research in non-Euclidean biomechanics" [7] etc. In other words, new daring attempts are made to build theoretical models of specific biological phenomena in the fields of morphogenesis, psychophysics etc. as formal theories of invariants of certain groups of transformations. In effect, the geometrization of physics goes hand-in-hand with attempts to geometrize biology. The theoretical biology of the future seems to be bound to become largely a group invariant biology. Natural sciences will then make a step to A. Eddington's ideal [8] of combining whatever we know of the physical world into one science whose laws could be expressed in geometric or quasigeometric terms.

For illustration of research along this line this article will describe the writer's research in higher symmetries and algorithms in self-structuring of living bodies. The research started with studies on the same kinds of symmetrical (Euclidean and non-Euclidean) algorithms for structuring the supramolecular organic forms in various lines and on various levels of the evolution of biological objects such as shells or molluscs and foraminifers, bodies of ring-shaped worms and miriapods, fish fins, bird features, flowers, leaves and offshoots. In the light of the Erlanger program, we will mean by the geometry of living bodies those geometrical groups of transformations whose properties dictate in supramolecular biological structures the standard integration of components into one whole.

Besides, symmetry of two figures implies that they can be made to coincide by some operation. Symmetry itself is a sum of two ideas, coincidence of figures and an operation which makes this coincidence possible. The former idea makes the concept of symmetry illustrative, whereas the latter is geometrically more meaningful, for from the group invariant, or geometrical, point of view far from every operation resulting in coincidence of some figures is admissible, only those are which belong to a "group of transformations", a stringent geometrical concept.

Studies of the laws and algorithms of organic shaping is a major direction in the biology of development which is expected by many to yield significant basic discoveries and important applications. Morphology has always been all important for biology but at the present time scientists in various disciplines give special attention to morphological self-organizations, properties of biological structures and their evolutionary transformations. Unlike, say, crystallography, mathematical biology does not rely on a generally acclaimed formal theory of morphogenesis, although numerous attempts have been made at creating such a theory with the aid of various initial control, engineering, diffusion reaction etc. models. Development of such a theory is difficult, largely because of a shortage of data on the common biological properties of morphogenesis which could be formalized and theoretically interpreted. R. Thom, a French researcher, was right when he said "... a geometrical attack of the morphogenesis problem is not merely justified, it is essential" [9]. The status of the mathematical biology of development is such that this science has yet to cover the evolution path from accumulation of knowledge on key morphological properties and adequate geometrical structures to the development of the desired theory, a path which has been covered by crystallography and other natural sciences dealing with objects much plainer than those of biology.

The very concept of symmetry emerged from ancient observations of the shapes of living bodies. We might witness this concept going full circle in a complex evolution of today's science of symmetry and taking up again its initial objects, living bodies and unraveling profound laws of living matter. This is all the more probable because, in the words of Weyl [3],

> "symmetry in the broad or narrow sense is the idea using which man has for centuries tried to obtain an insight into and create order, beauty and perfection."

The reader will certainly agree with V. I. Vernadsky [10] who said

> "Symmetry of living matter has not been thoroughly explored and its study remains a major task of the biologist.... This is the field of future tremendous fruitful endeavor."

2. THE HISTORY OF RESEARCH IN THE SYMMETRIES AND ALGORITHMS OF THE MORPHOLOGICAL SELF-ORGANIZATION IN BIOLOGICAL OBJECTS

> "(Biological) structures are only in particular cases dictated by the functions they perform while in a general case they obey some mathematical laws of harmony. The variety of shapes has its own orderliness, independent of the function, its own system which manifests itself, in particular, in symmetry detection from stringent mathematical description." (S. V. Meyen *et al.* [11])

Symmetry in the forms of biological bodies has always attracted the attention of natural scientists as one of the most remarkable and mysterious natural phenomena. The references of this article mention only a small fraction of the literature on this subject. Macromolecular symmetries were discussed by a special *Nobel Symposium* [12]. School curricula in biology include numerous instances of rotational, translational and mirror symmetries, and also symmetries of similarities of scale in biological bodies such as the walking gear of animals, flowers and offshoots of vegetation etc. (Let us note once more that the group of similarity transformations is the core of the Euclidean geometry and so such symmetries may be referred to as Euclidean as opposed to non-Euclidean symmetries which represent transformations from non-Euclidean groups.) A deeper biological insight resulted in discovery of new facts about very different biological bodies of various size and sophistication obeying the symmetry principle (and algorithmicity, which is worthy of special attention).

Biological symmetry is embodied to a larger or smaller degree in numerous biological theories, some highly controversial: N. I. Vavilov's law of homological series; A. G. Gurvich's theory of the morphogenetic field; V. I. Vernadsky's theory of the non-Euclidean geometry of living matter; the biological significance of the diffusion reaction model of morphogenesis, developed by A. M. Turing, and self-organizing growing automata, whose theory is being developed by J. von Neumann's followers; morphogenetic mechanisms behind numerous psychophysical phenomena including the esthetic preference of the morphogenetically significant golden section which is expressed by Fibonacci numbers etc.

We should not overlook the fact that from the geometrical (i.e. group invariant) viewpoint the entire classical biomorphology is essentially an extension of the group of similarity transformations. This is the case for morphological studies and theories of mirror symmetry and asymmetry of biological bodies, multicomponent biological forms which embody similarity symmetry according to A. V. Shubnikov [13], scaled three-dimensional growth of biological objects, dwarfs and giants among organisms of the same species etc. On the other hand, there is not good reason to believe that the geometrical fundamentals of morphology are confined to the similarity group; Vernadsky's assumption [10] on the important biomorphological significance of non-Euclidean geometry has been awaiting checking for a long time. Section 3 will discuss specific matters in non-Euclidean biomorphology.

More light must be shed on the geometrical fundamentals of morphology because they will dictate the geometric specifics of morphological studies and facts and their interpretation. A change of geometrical fundamentals would entail changes in all superior levels of this science and dictate new requirements and approaches to the development of formal theories which should be consistent with these fundamentals. Biomorphology is related to many fields of biology such as the biomechanics of postures, biomechanics of growing and motoric movements, psychophysics of perception etc. The updating of its geometrical fundamentals is capable of giving rise to new research approaches and encourage discovery of new properties of biological self-organizations in these fields.

In the existing variety of geometrically legitimate organic forms we will concentrate on structures whose components are integrated into an entirety in compliance with certain rules or algorithms which are the same along various lines and on various levels of biological evolutions. These structures, which may be referred to as algorithmical, are of special interest in theoretical morphology and related sciences such as biomechanics, biotechnology, bionics, synergetics etc. What is important is that in addition to regularly shaped biological objects there are some in which

the conjugation of components is less regular, if it exists at all. This paper will consider algorithmical supramolecular biostructures which are chains or manifolds decomposable ($S = S_1 \cup S_2 \cup S_3 \ldots \cup S_n \ldots$) into commensurable and regularly positioned elements (or motive units S_k). Figure 1 shows such manifolds discussed by the literature on biological symmetries. The general rule of representing decomposable manifolds is that the preceding motive unit is transferred

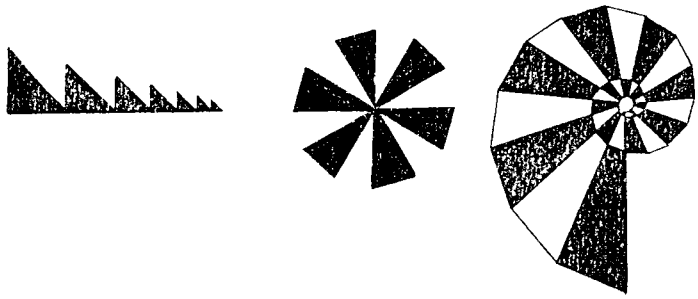

Fig. 1. Geometric examples of similarity cyclomeres or multiblock configurations with cyclic groups of similarity automorphisms (according to Ref. [13]).

into the succeeding one by a certain fixed similarity transformation g; in other words, the neighboring motive units S_k are mutually conjugated by an iterative algorithm

$$S_{k+1} = g * S_k. \tag{1}$$

Consequently, by reapplying the generating transformation g m times to a motive unit S_k a component S_{k+m} is obtained; mathematically speaking, in the set S_k a cyclic (semi-)group of transformations, G, is active which contains elements $g^0, g^1, g^2, \ldots, g^m, \ldots$ (a finite number of motive units in a biological object is neglected where necessary). In other words, this decomposition of the manifold thus organized includes a cyclic group of automorphisms and their motive units are aligned along the orbit of the appropriate cyclic group. For brevity, such configurations will be referred to as cyclomers, a term widespread in biology, no matter whether g is Euclidean or not in equation (1). Because Shubnikov [13] has made a significant contribution to research in similarity cyclomerism, the configurations shown in Fig. 1 are frequently referred to as Shubnikov symmetrical configurations. Section 3 will demonstrate the biological importance of non-Euclidean cyclomerism, which, thus far, has not been studied in biological bodies.

Similarity transformations in biomorphology are also known with reference to the scale of three-dimensional growth, which is fairly frequently observed in animals and vegetation over extensive periods of individual development and is accompanied with mutually coordinated growth behavior of small zones distributed in the volume of the body—a behavior which is geometrically described as a scaling transformation. With the transformation of as few as three points of the growing configuration known, transformation of the continuum of its points may be assessed. Note that the growth-related ability of living organisms of a variety of species to exist in morphologically identical modifications on various scales is probably the morphological property of living matter that man has been aware of for the longest time and has been reflected in scientific, mythological and fiction literature in which dwarf–giant relationships are variously described.

Do the Shubnikov similarity symmetry and the scale of the volume growth exhaust all geometrically legitimate kinds of mutual conjugation of parts in a structure and ontogenetic transformations in living bodies? Or do they act in biomorphology as very particular cases of the kinds which are built around non-Euclidean groups of transformations containing a similarity subgroup? The writer's research has provided a positive answer to this latter question. As S. Lie, a classical mathematical writer, noted [14], there are two basic ways to extend the similarity transformation group, either to Möbius transformations or to projective transformations (see the Appendix). Now let us proceed to new findings on non-Euclidean symmetries in mutual integrations of individual biological bodies.

3. NON-EUCLIDEAN GEOMETRIES AND ITERATIVE ALGORITHMS IN ORGANIC FORMS

"If we do not succeed in solving a mathematical problem, the reason frequently consists in our failure to recognize the more general standpoint from which the problem before us appears only as a single link in a chain of related problems. After finding this standpoint, not only is this problem frequently more accessible to our investigation, but at the same time we come into possession of a method which is applicable also to related problems." (D. Hilbert [15])

The biological value of similarity cyclomerism (Fig. 1) seems to be clear. What configurations form if the generating transformation g of equation (1) is a Möbius or affine transformation? And do such non-Euclidean cyclomerisms have biological analogs? Analysis reveals that the manifold of cyclomeric configurations noticeably expands in this case and includes, in addition to cylindrical, conic and helical forms associated with similarity cyclomerism, more complex configurations such as lyre-, sickle- bud-shaped etc. (Fig. 2). In these configurations the motive units may be different in shape and the variation of these units along the cyclomerism may be essentially non-monotone. What is important is that non-Euclidean cyclomerisms are as widespread in biological bodies along various lines and at various levels of evolution as the similarity cyclomerism, but in the science of biosymmetry they have not been studied. Remarkably, non-Euclidean and Euclidean cyclomerism are observed in analogous multicomponent biological bodies simultaneously.

In particular, the horns of numerous animals are helically or rectilinearly conical and thus can be described as similarity cyclomerism. In other animals the horns are essentially different and configured as non-Euclidean cyclomerism. Thus, Fig. 3 shows the horns of a *Pantholops hodgsoni* [16] which is described by a cyclomerism obtained by a Möbius generating transformation (of the so-called loxodromic type).

This example illustrates the general morphological procedure which reproduces the conventional procedure in which similarity cyclomerisms are analyzed. To begin with, the manifold of basic geometrical configurations is obtained, for instance by computer graphics. These configurations

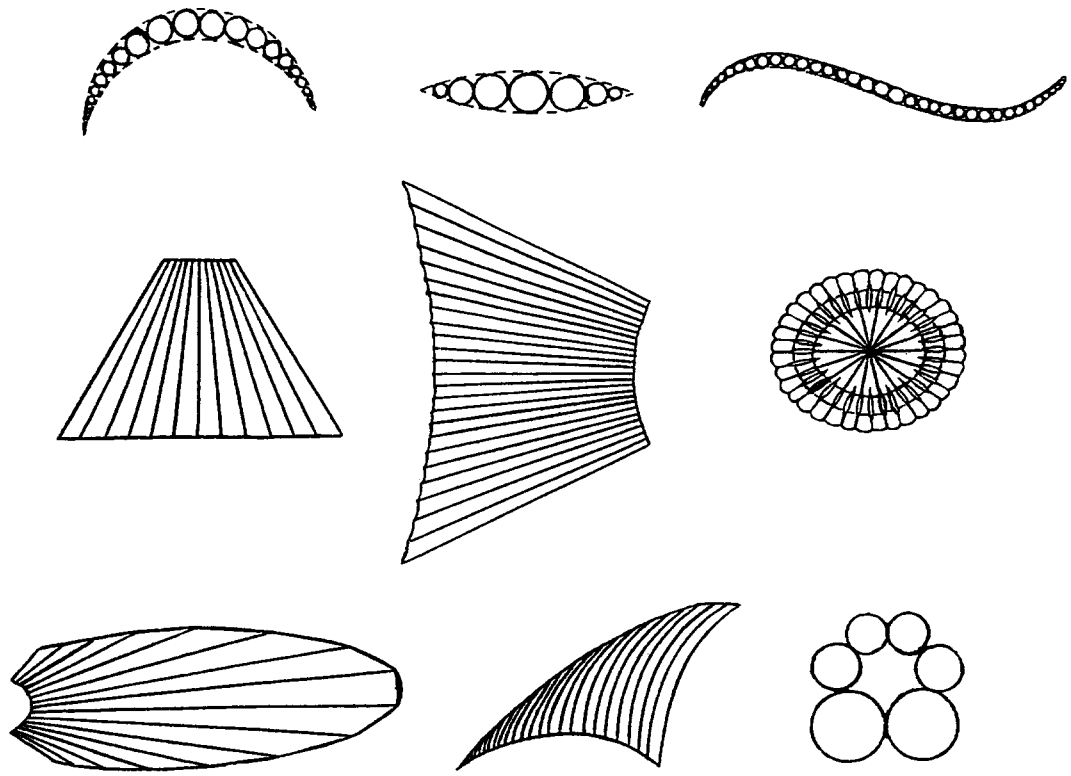

Fig. 2. Examples of Möbius, affine and projective cyclomeres.

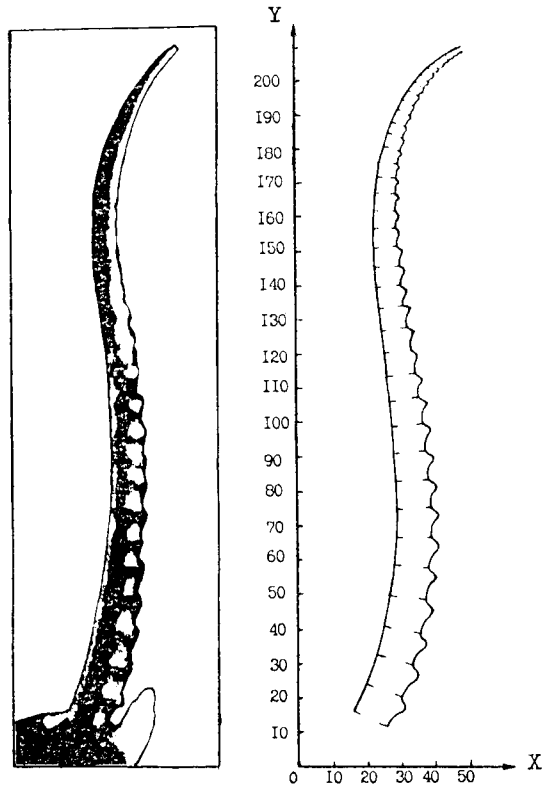

Fig. 3. A segmented horn of the *Pantholops hodgsoni* orongo [16] and its model as a cyclomerism with a Möbius generating transformation.

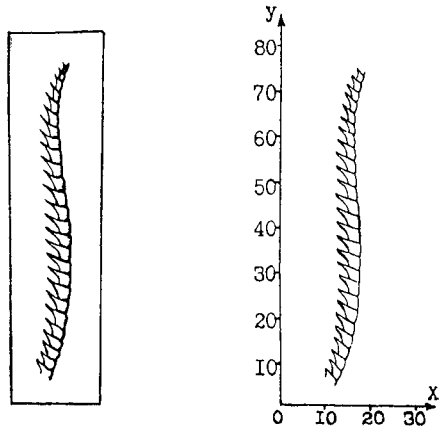

Fig. 4. A comb-like antenna of an insect and its model as a Möbius cylcomere.

may be obtained by applying the iterative algorithm (1) to some motive unit, in particular, a point, with g from the group G of, say, Möbius or projective transformations. These abstract configurations and actual biological structures are compared. In establishing visual kinship the coefficients of g for this biostructure are updated in the following procedure: the general analytical form of transformation from G is written down (see the Appendix) and its coefficients are computed by substituting in this general equation the coordinate of those associated points of motive biological units which transform into one another by this transformation. Applying the transformation of g thus specified to the motive unit the required number of times the desired cyclomerism is obtained which models the biostructure as a whole.

Non-Euclidean cyclomerisms are also observed in the antennas of insects (Fig. 4). Figure 5 shows that the sequence of vertebral disks in the human dorsal vertebra is reproduced by Möbius cyclomerisms, in contrast to the configuration of the spine of numerous animals, in particular lizards, which is described by conventional similarity cyclomerism. By modifying the generating transformation, and thus bending the "normal" configuration of the spinal cord cyclomerism, models of morphogenetic anomalies of the spine can be obtained. This is consistent with our data on Euclidean and non-Euclidean cyclomerisms, in man, animals and vegetation, both in normal and pathological shapes. This also agrees with the formula of the morphology of plants that ugliness is a version of the norm and that, as S. V. Meyen, a prominent Soviet researcher, wrote in 1973,

> "The most important peculiarity of deviating forms from the structural viewpoint is in their approximation, by means of the deviating character, to another (often closely related) taxon, where this character state is a normal one. This phenomenon was named by Krenke as 'the law of related deviations'." [18]

Möbius, as well as Euclidean, cyclomerisms are to be found in the configuration of bird feathers, for in addition to straight or helical shapes there are lyre-shaped feathers, as in the case of lyrebirds—so named because of the shape of their tails—or the tail of the Caucasian heathcock.

Möbius cyclomerisms of the lyre-shaped (or loxodromic) type are also observed in the structure of ambulacrum fields in sea urchins, discontinuous lateral generating lines in the helical shells of some molluscs, the long horn-like growth in the beak of the *Chasmorynchus niveus* bell-bird, sculptural lines in the body of the *Hippocampus guttulatus microstephamus* etc. Similarity cyclomerisms are most vivid in the shells of protozoa, but non-Euclidean cyclomerisms are also visible there (Fig. 6).

Both kinds of cyclomerisms are also observed in the structure and functioning of the vestibular organ and the eye muscles; the structure of the nervous system, bone tissue, vessels and muscles; positioning of biologically active points in the human body; cyclomeric buildings instinctively made by protozoa and social insects; paths of human and animal motions in normal and extraordinary conditions; formation of psychophysical delusions; the esthetics of proportions and shapes in architecture and art etc. They are manifest in unicellular as well as multicellular organisms; consequently, the cell is not a morphogenesis unit in a general case. Figures 7–14 show other cases of the variety of non-Euclidean cyclomerisms in living organisms.

Depending on the generating transformation, biological cyclomerisms may be explicitly discontinuous or practically continuous. Iterative algorithms are functionally useful in the generation of multicomponent biological structures because of the obvious convolution of information in the genetic encoding and morphogenetic implementation of such structures. In a living organism a broad range of cyclomerisms are usually present, in addition to morphological structures of a less regular nature. Furthermore, the cyclomeric structuring is reflected in the coloring series, weight parameters and numerous other characteristics of living organisms, as well as in the series of body units.

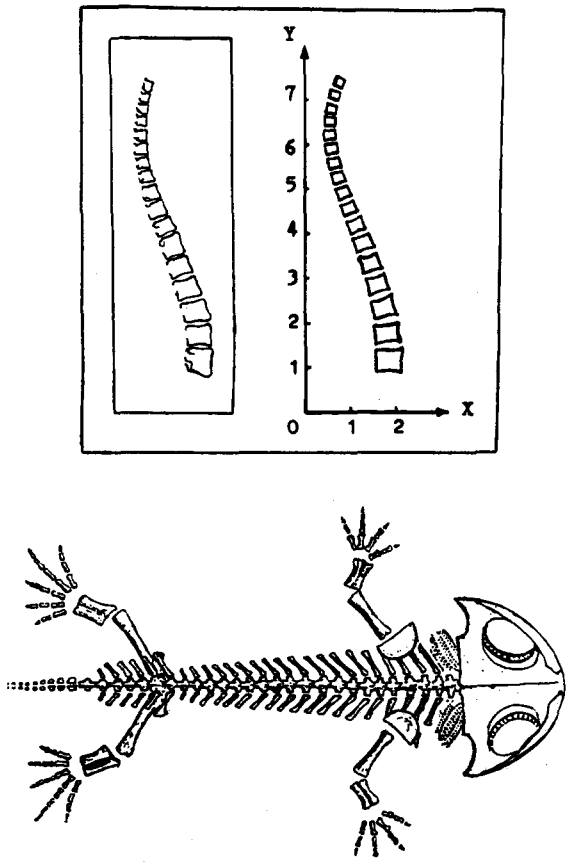

Fig. 5. Vertebra of the torso part of the human spine [17] and its model as a Möbius cyclomere. For comparison a similarity cylcomere in the structure of the torso part of the spine is shown [17].

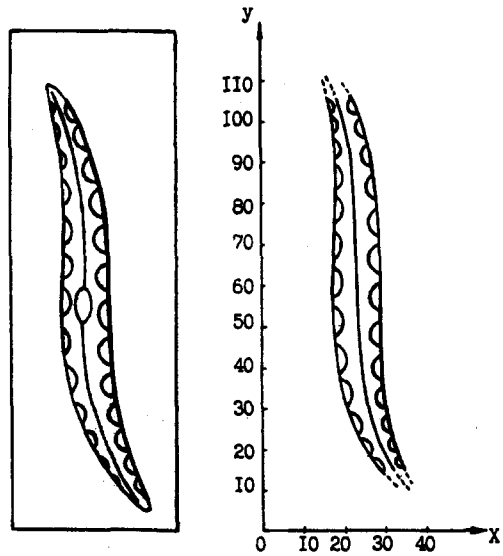

Fig. 6. The shell [19] and its model as a Möbius cyclomerism.

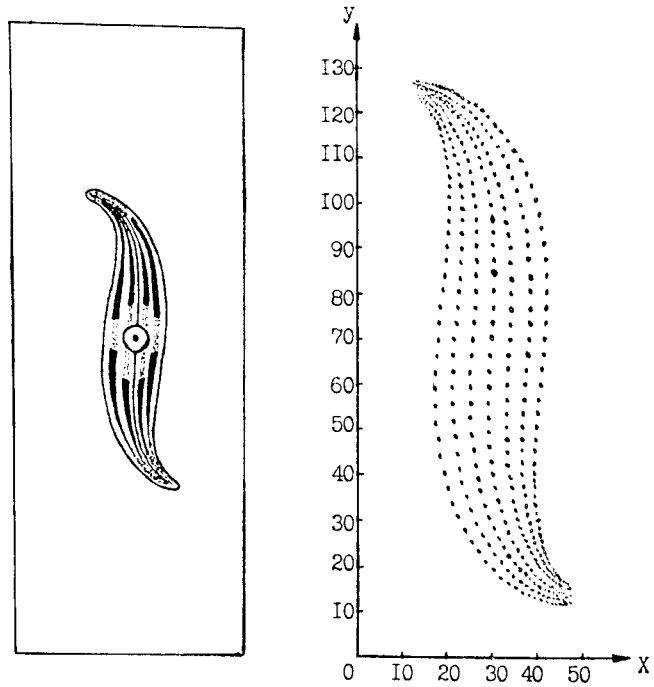

Fig. 7. Cyclomeric structures in unicellular organisms. *Left*: *Naviculla hippocampus* [17]. *Right*: a model cyclomeric structure whose lyre-shaped forming line is the orbit of a point in a Möbius loxodromic transformation; a sequence of the orbits makes a cyclomerism with other Möbius generating transformations.

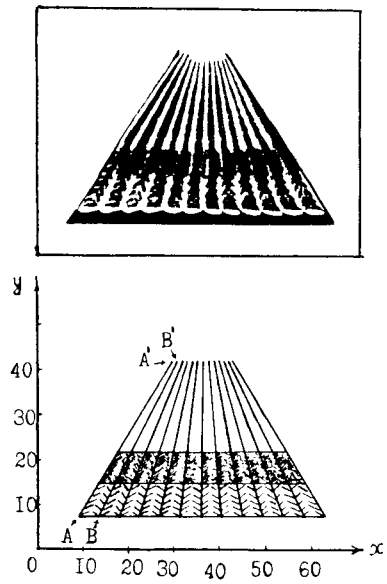

Fig. 8. The multiblock structure of the scale of a bone fish [19] and its model as a cyclomerism with an affine generating transformation $X_{k+1} = X_k - 0.099\ Y_k + 5.24$; $Y_{k+1} = Y_k$. ABB'A' is a specific cyclomerism.

The non-linearity of generating Möbius and projective transformations makes it possible for the size of individual cyclomerisms and their relations to vary in the same cyclomerism series; the geometrical invariants over the entire series are only those characteristics that are invariant with projective (or Möbius) transformations, e.g. the values of cross ratios or wurfs. Because Möbius

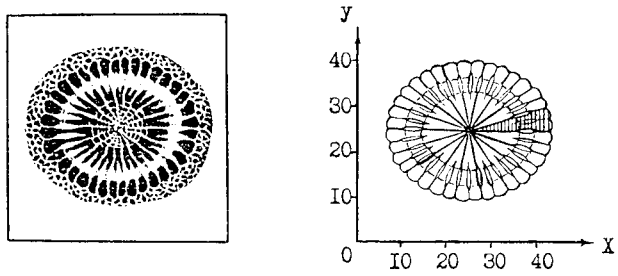

Fig. 9. A multiblock part of a four-ray *Tetracoralla* [19] and its geometrical model as a cyclomerism with a generating affine transformation $X_{k+1} = 0.94\ X_k - 0.4\ Y_k + 11.48$; $Y_{k+1} = 0.29\ X_k + 0.94\ Y_k - 5.82$; which transforms every radial segment (one of which is shaded for illustration) into its neighbor.

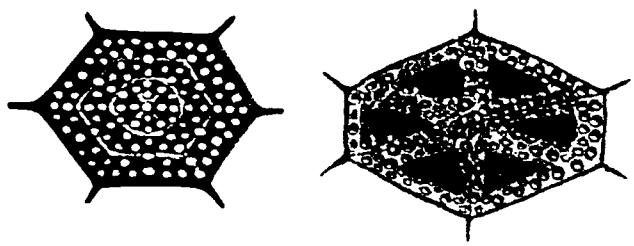

Fig. 10. Bone plates of the *Ostractones* [19] test with a configuration of affine cyclomerism.

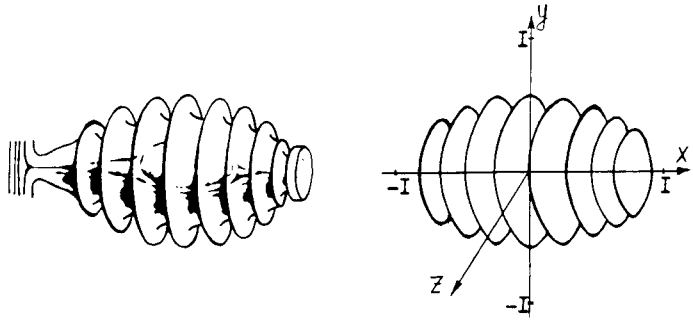

Fig. 11. An individual sexual capsule of *Eusertularia exserta* of the hydroid class [19] and its model as a cyclomerism with a projective generating transformation $X_{k+1} = (X_k + 0.258):(0.258\,X_k + 1)$; $Y_{k+1} = 0.966\,Y_k:(0.258\,X_k + 1)$; $Z_{k+1} = 0.966\,Z_k:(0.258\,X_k + 1)$.

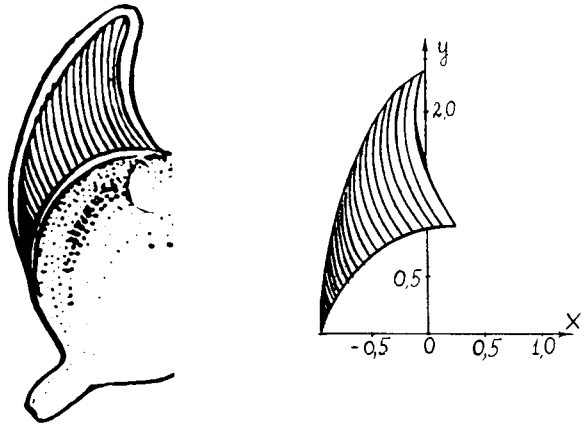

Fig. 12. The multiblock structure of the *Phylospadix scouleri* fruit [20] and its model as a cyclomerism with a projective generating transformation $X_{k+1} = -(X_k + 0.08):(0.08\,X_k + 1)$; $Y_{k+1} = 0.997\,Y_k:(0.08\,X_k + 1)$.

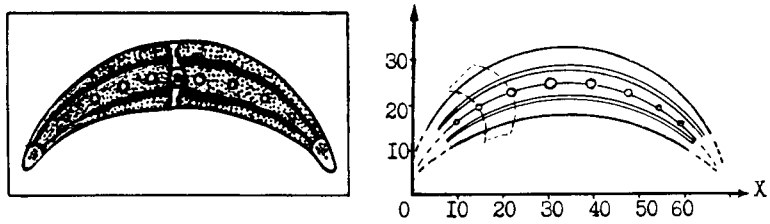

Fig. 13. The desmidean *Glosterium lebleinii* alga [20] and its geometrical model as a cyclomerism with a Möbius generating transformation of the hyperbolic type (the dashed square shows a cyclomerism variety).

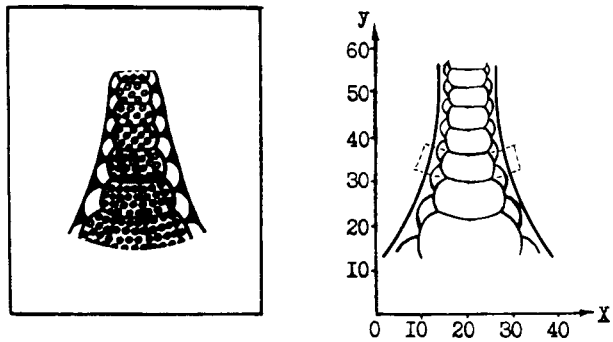

Fig. 14. A multiblock structure in the skeleton of the *Stichocapsinae radiolaria* [21] and its geometrical model as a cyclomerism with a Möbius generating transformation of the elliptical type (the dashed square shows a cyclomerism variety).

and projective transformations underlie the conformational and projective geometries, these abstract geometries materialize in biological structures.

Until recently the desirability and adequacy of non-Euclidean transformations in the morphology of living bodies remained an open question. Furthermore, numerous scientists believed that the Euclidean transformations were quite sufficient for biomorphology. In particular, J. Bernal, a well-known British researcher, and S. Carlyle used the Euclidean group of movements in their concept of generalized crystallography which was to cover, in particular, the symmetries of living bodies [22]. The complexity, importance and shortage of knowledge of the geometry of living bodies were summarized by Yu. A. Urmantsev, a Soviet student of biosymmetries, in 1971:

> "It would be no exaggeration to say that the geometrician who succeeds in developing the geometry not of the entire living natural but of a 'mere' flower will be worthy of rewards for both boldness and genius." [23].

The desire to use generalized geometric concepts in biology was felt as long ago as 1942, as in papers by D'Arcy W. Thompson [24], D. V. Nalivkin [25] and V. I. Vernadsky [10]. The research reported in this article has certain features which make it different. D'Arcy Thompson tried to find transformations which make one another into bodily shapes of various organisms such as perch and pike. He obtained, occasionally, very complicated curvilinear transformations which reflected, in particular, the relative autonomy of the morphological development of body organs; he did not use the Erlanger program as the basis in this comparative analysis. Nalivkin introduced curvilinear symmetries without reference to the Erlanger program; the specific rules that he proposed for construction of "symmetrical" transformations result in transformations which do not add up to groups of point transformations, because they disturb the group principle of the one-to-one correspondence of points and are not conventional in other fields of natural sciences. Vernadsky [10] did not make his views on the non-Euclidean geometry of living matter explicit and did not rely on the Erlanger program either. None of these papers considered special non-Euclidean iterative algorithms and the configurations they lead to with cyclic groups of non-Euclidean automorphisms. Unlike those papers, this article analyzes, above all, the rules of mutual conjugation of natural components in the individual biological body; this analysis proceeds along the lines of the Erlanger program, important for mathematical natural sciences, in terms of specific non-Euclidean groups of transformations which include the similarity subgroup; iterative algorithms which gave rise to discontinuous configurations with a cyclic (semi) group of non-Euclidean automorphisms are given special attention.

4. EUCLIDEAN AND NON-EUCLIDEAN ITERATIVE ALGORITHMS IN THE KINEMATICS OF BIOLOGICAL MOVEMENTS AND PHYSIOLOGICALLY NORMAL POSTURES

> "The movement is in many respects similar to an organ ... seems to be a very useful idea, notably as far as a stable and universal movement such as locomotion is concerned." (N. A. Bernstein [26])

The above discussion dealt with the structure of static organic forms. But cyclomerisms also have a direct bearing on the kinematics of a broad range of biological movements which can, on numerous occasions, be interpreted as process in which cyclomerisms replace one another.

Growth transformations of numerous metamerically-structured biological bodies (larvae of insects, bodies of centipedes and worms etc.) in which metamery with one metameric step is taken over in certain ages by a metamery with another step is a trivial example of such transformations.

Less trivial cases are mutual replacements of cyclomerisms of different kinds, such as translational cyclomerism replaced by rotational cyclomerisms. This case is represented in Fig. 15 via the stages of colony development in phytomonadids (according to Ref. [27]), where, in addition to various rotational cyclomerisms, translational cyclomerisms are involved. In addition, the number of motive units changes with the age.

When non-Euclidean cyclomerisms are brought into the picture, a better insight is obtained into the relation of the development of organisms or their parts with the ability of the living matter to stay in different cyclomerism states. For example, in the development of the salamander fetus the sequence of somites which is described at an early stage by the similarity cyclomerism as a logarithmic helix changes into a lyre-shaped Möbius cyclomerism.

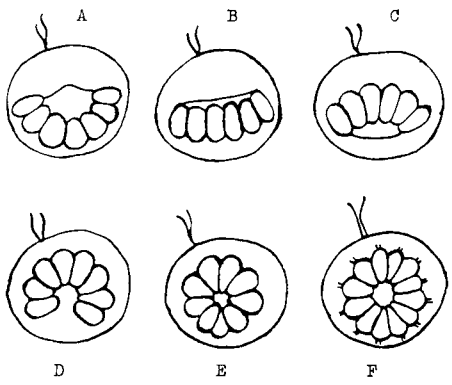

Fig. 15. Development of a phitomonadid colony [27]. A–E are excurvation stages in *Eudorina illinoisensis*.

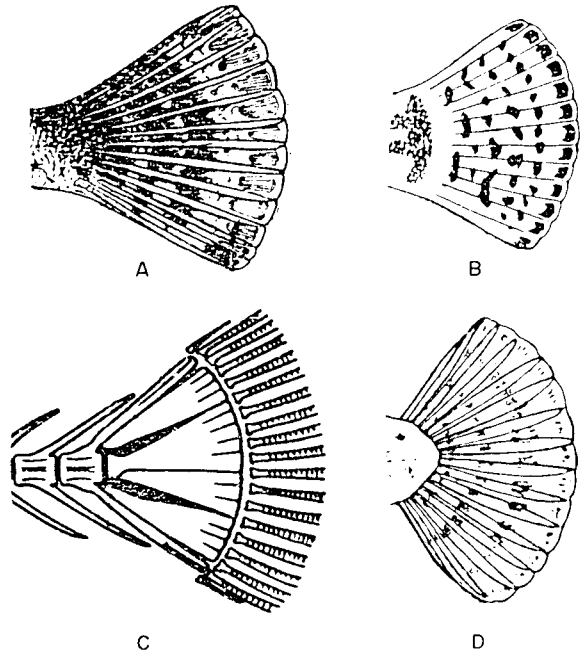

Fig. 16. Caudal fins in *Scorpaena porcus* Linne (A), *Siniperca chua-tsi* (B), *Platessa platessa* (C) and *Rhombus maeoticus* (D) [28, 29] as cases of similarity cyclomerism.

In studying the biological cyclomerisms, the following theorem is useful. If a cyclomerism with a generating transformation g (such as similarity) is subjected to, say Möbius or projective, transformation S, the newly formed manifold with decomposition is again a cyclomerism with a generating transformation P:

$$P = S \cdot g \cdot S^{-1}. \tag{2}$$

By this theorem, with the generating transformation of the cyclomerism forerunner known, one can determine the generating transformation of the cyclomerism image which is obtained by an arbitrary transformation S. The form of the motive unit of the transformed cyclomerism is a transformed associated unit of the forerunner changed by S.

In analogy with the polymorphism of crystals whose lattices can, under certain conditions, be restructured with a change in their symmetry groups, the ability of living bodies to restructure their Euclidean and non-Euclidean cyclomerisms may be referred to as cyclomeric polymorphism. The concept of cyclomeric polymorphism sheds additional light on the fact that in individual development of multicomponent structures of the cyclomeric type, the transition from one cyclomerism to another proceeds as a relay race in the series of motive units (e.g. opening of cones as the scales ripen, of composite flowers such as daisies etc.) and may be referred to as the cyclomerism change wave. The natural morphogenetic movements in numerous multicomponent biological bodies such as the so-called excurvature in the development of the Volvox colony may be modeled as the simultaneous propagation of two or more cyclomeric change waves in the series with a fixed interval.

Euclidean and non-Euclidean cyclomerisms are involved in the apparatus of motoric movements in numerous animal organisms, in particular in the multineedle fish fins. Classical cases of similarity cyclomerisms in the tails of some fish are shown in Fig. 16; and cases of affine and projective cyclomerism in the tails of others, in Figs 17 and 18.

In the spirit of the above reasoning, numerous specifics of the structure of fins as multicomponent parts may be described and explained in terms of symmetric morphogenesis algorithms, leaving the locomotoric functions of these organs aside. Incidentally, the superficial but widespread view is that the fins are intended for swimming and that evolution made their structure optimal for this function and so the specifics of the structure may and must be derived from hydrodynamics. In

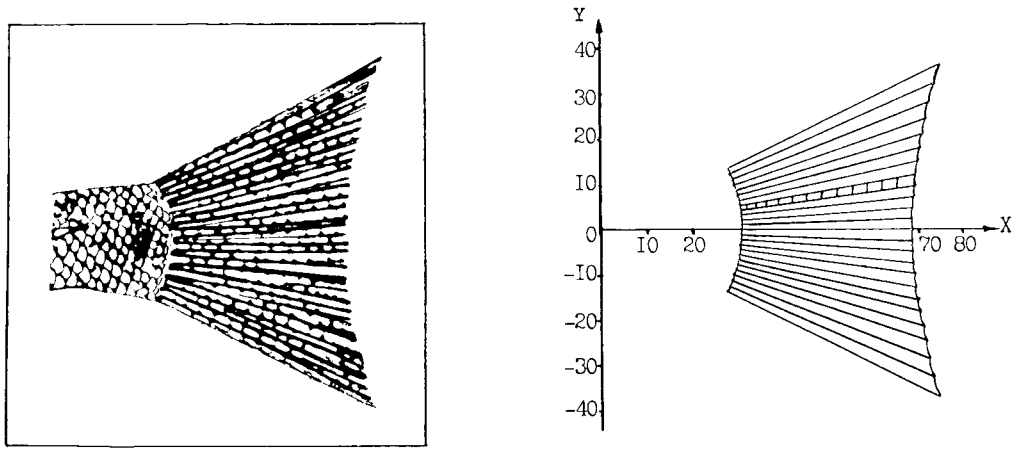

Fig. 17. The multiblock structure of the caudal fin in *Acanthurus* sp. [29] and its model as a cyclomere with a projective generating transformation $X_{k+1} = (0.998\ X_k - 0.087\ Y_k):(-0.0016\ Y_k + 1)$; $Y_{k+1} = (0.0347\ X_k + 0.9985\ Y_k):(-0.0061\ Y_k + 1)$ (one motive is shaded for illustration).

other words, knowledge of hydrodynamic equations is supposed to be sufficient for understanding the structure of fish fins. The above findings of group invariant analysis refute this view and draw attention to the multitude of fin functions, which include, in addition to the locomotoric function, the general biological function of contributing to the morphogenetic processes of inheriting the body form with algorithmical mutual conjugation of components.

This remark leads to a question, to what extent the structure of a living organism and the performance of some function in the environment may be taken by machine designers as a model to be imitated, bearing in mind the millions of years of evolution and natural selection. Studies of symmetry mechanisms in biological bodies reveal that far from everything in the structure of organs is dictated by optimal adaptation to specific functions in the environment by the laws of biological morphogenesis are very important. In effect, the machine designer can concentrate on optimal functioning in the environment and does not have to adapt to the needs of the internal requirements of the organism and purely biological requirements (volume growth in the individual development, inheritance of biological properties etc.). Still, the achievements of nature in the performance of functions must not be dismissed offhand. Nature itself has long solved numerous problems of functioning in the environment that designers face today. These solutions are not necessarily optimal in the usual sense but they are invaluable because they highlight the existence of challenges and demonstrate ways to tackle them; they stimulate human imagination and have acted as catalysts of technological progress in the entire course of human history.

Let us adopt a physiologically normal posture, a concept widely used in biology. In the entire set of postures or positions of its parts, such as the tail, the trunk etc., some positions of the body components *vis-à-vis* one another are inherited. They are instinctively adopted in a stereotype way in cases of fear, weariness, rest etc. They include the posture of rest of the starfish with symmetrical

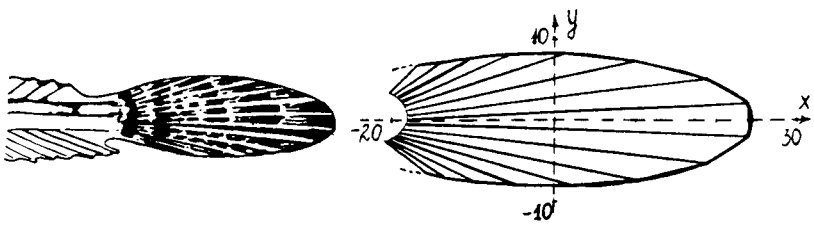

Fig. 18. The caudal fin in *Lumpenus lampetraeformis* [29] and its model as a cyclomerism with a projective generating transformation

$$X_{k+1} = \frac{0.83\ X_k + 1.27\ Y_k - 2.26}{-0.05\ Y_k + 1}; \qquad Y_{k+1} = \frac{-0.14\ X_k + 0.91\ Y_k - 3.26}{-0.05\ Y_k + 1}.$$

Fig. 19. Physiologically normal postures of the elephant trunk configured as similarity cyclomerisms (a, b) and a loxodromic Möbius cyclomerism (c).

positioning of the rays which are described as rotational cyclomerisms, the mutual positioning of components of the fin at rest, the metameric posture of a resting caterpillar etc.

Research into such in-born postures and positions of groups of mobile components ("segmentary" postures) of the support and motor apparatus is also important because they appear to be the references for construction of motoric movements and for the complex system of muscular drives with which genetically dictated characteristic conditioned reflexes are developed, the analyzers are coordinated etc. A better insight into their relations with the functioning of various system organisms such as neuromuscular, vestibular etc. will be useful in optimizing human postures and movements; reduction of muscular weariness in human operators; reduction of the detrimental effects of unfavorable factors such as vibrations and overloads; increasing the vestibular stability; choice of optimal postures in the cases of protracted immobility, for instance in treating broken extremities; forecasting and explaining human senso-motoric reactions under unusual conditions such as zero gravity in space missions; better coordination of space suits and exoskeletons with the specifics of the human support and motor apparatus; improving the movements of athletes; improving sporting equipment; and the development of senso-motoric systems in zoomorphous robots.

The author's research has revealed that the set of inherited postures includes, in addition to (segmentary) postures described by similarity cyclomerisms, postures which are described as non-Euclidean cyclomerisms, which significantly expands the data on the relationship of such postures with symmetrical morphogenesis algorithms. Besides, the kinematics of natural motoric movements (in analogy with the above morphogenetic movements) is often a transition from one cyclomerism type to another, or involves cyclomeric polymorphism.

In particular, the tubular *Glomeris romana*, whose segmented body is usually rectilinearly elongated and is metamerically configurated is known to convolve into a segmented ring; this movement is described as the transition from one similarity cyclomerism to another.

Euclidean and non-Euclidean cyclomerisms describe the stereotype postures of the elephant's trunk, in which it is straight, coiled into a logarithmical helix or is lyre-shaped when the elephant blows it (Fig. 19). The lyre-shaped Möbius cyclomerism is characteristic of the attacking posture of the cobra whose body is obviously segmented and the configuration of the frightened rat's tail (Fig. 20). A kindred cyclomeric shape is also observed in stereotype postures of the multivertebrum necks of some birds such as flamingos and swans. The proboscis of many butterfly species, which

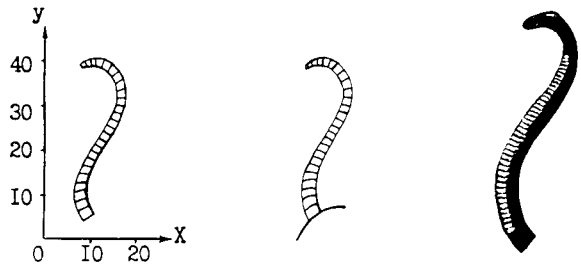

Fig. 20. Möbius cyclomeres in physiologically normal postures in animals. *Left*: a cyclomere with a Möbius generating transformation. *Middle*: posture of the tail in a frightened rat. *Right*: the posture of a cobra before attack (*photo*: S. I. Bortkevich).

is coiled into a logarithmic helix at rest, straightens when used and this movement is described in a natural way as the transition from one cyclomerism state to another. Similar symmetric algorithmical specifics of movements are also observed in some plants which are between animals and vegetation with respect to mobility. In particular, in response to exogenous irritation the branch of the sensitive mimosa, whose leaves are cyclomerically positioned, can fold its leaves along the branch but this new configuration of the sequence of leaves is also cyclomeric. The kinematics of various motoric movements such as walking and crawling in numerous animal organisms such as multipedea, caterpillars and ring-shaped worms can also be modeled as the transition from one cyclomerism to another, and waves of cyclomerism changes propagating along a multicomponent body.

The similarity of growth and motoric movements and their relationship with shaping in terms of cyclic groups of authomorphisms confirms that the organization of biological kinematics cannot generally be considered separately from the morphogenesis principles (which reflect the conditions in the internal environment of the organism and the self-structuring ability of this environment) or derived solely from the classical mechanics laws and the functional purpose of the movement, as some still try to do.

The specifics of self-conjugation of organic macrocomponents in the course of self-assembly in the Euclidean, Möbius and projective types of cyclomerism is worthy of special attention. As the literature noted long ago, supramolecular organic bodies are different from crystals, in particular, in the curvilinearity of their surface and in that their mutual conjugation into orderly ensembles does not generally occur by the crystallographic rule of the densest packing possible. In the biological self-assembly of individual bodies various reactive chemical groupings nonuniformly distributed over the surface of these bodies and generating surface energy are very important. The biological self-assembly is consistent with the well-known rule of stability in a minimal energy state, for the appropriate chemical groupings on conjugated surfaces react and release energy and the total surface energy of the system is less than the sum of the surface energies of individual organic surfaces.

What is important is that, by all accounts, a significant role in the mutual conjugation of curvilinear organic surfaces during self-assembly is played by surface lines having extreme mechanical and geometrical properties; it is along these lines that chemical groupings assemble (and entire ensembles of metabolic systems including groups of cells), which make possible the accretion of adjacent surfaces. To put it differently, biology, unlike conventional crystallography, may and must deal with the folding of conjugating adjacent curvilinear surfaces along special lines and zones which reflect, in particular, the properties of the surfaces' curvature and the tension distribution on the surface. For certain reasons surface curvature and asymptotic lines can be classified with the above special conjugation lines. Recall that a surface curve is referred to as an asymptotic line if its direction at every point coincides with that in which the normal curvature of the surface is equal to zero. A surface curve is referred to as a curvature line if its direction at every point coincides with that of the extremal value of normal curvature. Note that surface curvature (asymptotic) lines are invariants of Möbius (projective) transformations.

If mutual conjugation of adjacent organic surfaces proceeds along lines invariant with a certain group of transformations, then with a biological transformation (in particular, in the course of volume growth) of conjugated surfaces by a transformation from that group, new surfaces are obviously conjugated along the same types of surface lines and there is no need to move the chemical groupings which make the accretion possible to a significantly new relative position on every surface. This fact is attributable to the probable biological desirability of using certain groups of transformations in the mutual conjugation of numerous surfaces into one biological body along surface lines invariant with these transformations. From the proposed point of view, the case of Möbius transformations is associated with the positioning of surfaces with respect to one another along their curvature links; in the case of projective transformations, along asymptotic surface lines; in the case of similarity transformations, along curvature and asymptotic lines simultaneously (because the group of similarity transformations is an intersection of the groups of Möbius and projective transformations).

There is abundant evidence in favor of a morphogenetic significance of these lines. In particular, the surface curvature at a specific point is by no means significant morphogenetically. Curvature

and asymptotic lines largely reflect the mechanics of hulls. For instance, in the case of thin (momentless) hulls for a broad range of surfaces (axial symmetrical surfaces with an axial symmetrical load) the surface curvature lines coincidence with the lines of their main tensions; in other words, curvature lines are identified by the extreme mechanical properties. In biology the latter fact justifies the positioning along these lines of the centers of chemical interaction between the organic surface and the environment, because the extreme mechanical tensions probably have an extreme impact on the opening of micropores; deformations of structural elements of chemical groupings on the surface; renaturation and denaturation of collagen molecules, morphogenetically important; biological rhythms in morphogenetic processes etc.

The author has developed a few models which represent the formation of Euclidean and non-Euclidean biological cyclomerisms with the assumption that replications of tissue layers are morphogenetically significant (valuable data on possible layer replication in various substances have been obtained by Weiss [30]).

5. THE KINEMATICS OF THE THREE-DIMENSIONAL GROWTH OF BIOLOGICAL BODIES WHOSE PARTS ARE NOT CYCLOMERICALLY POSITIONED

"Wisdom is rooted in watching with affection the way people grow." (Confucius, quoted from Ref. [31, p. 7]).

We have thus far considered the kinematics of three-dimensional biological growth with reference to cyclomeric bodies, in which it is described as the interchange of cyclomerisms (leaving out for a time the change in the number of motive units). Numerous biological bodies are, however, characterized by unorderly or complicated positioning of their parts and can be treated, at best, as ensembles of numerous cyclomerisms rather than a single cyclomerism. Nevertheless, these biological bodies are also capable of integral three-dimensional growth of geometrically consistent types.

The three-dimensional growth of living bodies is a challenging and mysterious case of the orderly cooperative behavior of numerous elements. This growth is essentially different from the surface growth of crystals, which occurs by accumulation of matter on the surface and does not involve the internal areas. The three-dimensional growth of living bodies entails change in the dimensions and frequently in the shape of the internal areas. The process is cooperative in the sense that by knowledge of the transformation of some points in the body figure, the transformation of the entire set of points in that figure may be determined.

This is the case for vegetation and animals (growth of leaves and flowers of some plants, larvae of insects, adult fish etc.) and is expressed in the proportional growth of all parts of the body; with the transformation of the position of three points in a body undergoing a known scaling transformation, the transformation of the entire set of points in the body can be determined.

The writer's research has revealed the existence of non-Euclidean kinds of three-dimensional growth not previously known, notably Möbius and affine. Let us take a closer look at the former.

If the three-dimensional growth is modeled using the concept of some active medium which grows through coordinated growth of every point-like zone of its volume, then the scaled three-dimensional growth occurs in this model if two conditions are observed: (1) scaling of every local zone of the body, or growth of every point-like zone with equal intensity in all directions (locally isotropic growth); (2) equality of the scales on which all local zones of the scaling body change. These local conditions may be regarded as postulates from which the specifics of scaling the "entire" body are derived.

But does nature not employ simpler types of growth changes which satisfy a simpler list of conditions, for instance containing only the first condition? Let us see what the corollaries would be of the local isotropic growth, a condition very natural physcially and one of the plainest local rules. Geometry states that with this type of local changes, in which every local zone is subjected to local transformations, the shape of the "entire" body may change dramatically from the layman's point of view because similarity "in the small" does not imply similarity "on the whole" in the general case (which is under consideration), where similarity transformations are not identical in different local zones. Transformations under which every local zone of the body

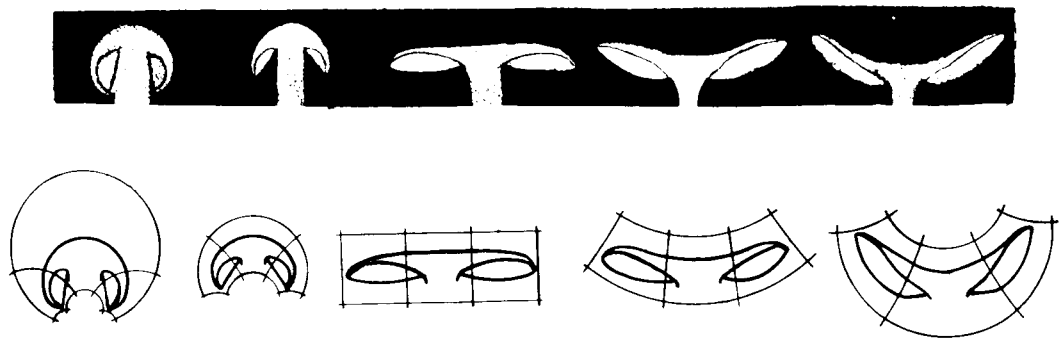

Fig. 21. Growth transformations in the cap of the fly-agaric *Amanita* and their modeling as Möbius transformations.

undergoes similarity transformations are referred to as conformal. (It was in terms of local similarity that Gauss developed a theory of conformal transformations.) For the case of a three-dimensional space, to which this discussion will be confined, all conformal transformations are known to be covered by the group of Möbius transformations.

Let us proceed to specific examples of adequate modeling of the three-dimensional growth of biological bodies in terms of Möbius transformations. As in the case of cyclomerism, Euclidean and non-Euclidean symmetrical transformations are carried out in analogous biological bodies. Thus, the caps of numerous kinds of mushrooms, such as *Lactarius vellereus* and *Morchella culenta*, scale up in their individual growth. On the other hand, in some mushrooms the cap transformation over an extensive period is described as Möbius. This case is illustrated in Fig. 21 for the fly-agaric Amanita which changes the cap shape. Similar changes in shape also occur in the caps of field mushrooms.

Möbius transformations are applicable also to pathological deformaties as well as normal growth. This also confirms that in normal and pathological self-formation of biological bodies kindred mechanisms and symmetry rules are involved.

Scaled and Möbius kinds of three-dimensional growth are also observed in the individual development of composite flowers, fruits etc. Because three-dimensional scaling also occurs in animals, we tried to describe nonlinear transformations occurring in the human skull with aging as Möbius (Fig. 22) and succeeded in obtaining a first approximation.

The above results of group invariant analysis legitimize the use of new tools of morphometric analysis, invariants of conformal and projective geometries, such as the wurfs mentioned above, in the research of biological structures. (Note that the founder of today's projective geometry J. Desargues, French architect and engineer (1593–1662), widely employed biological terminology, probably assuming a kinship between the projective structures and nature.) Some of these tools may be used in the construction of geometrical models of numerous organic bodies from multistaged symmetrical units, for the body is made of symmetrical (in the sense of, say, Möbius symmetry) blocks of the first order combined into symmetrical blocks of the second order etc. In particular, this principle manifests itself in the kinematics of the human body, where the mirror symmetry of the two halves of the body, which act as second-order blocks, is supplemented by an approximate Möbius† symmetry of the longitudinal proportions of the three-component kinematic blocks which comprise this structure, viz. the phalanxes of fingers, the three-membered extremities (shoulder–forearm–wrist and hip–shin–foot) and the three-membered body (in anthropology the body is subdivided into the upper, torso and lower parts). This is the conclusion obtained from the computation of the wurfs of these three-membered blocks from the anthropometric data of

†In the unidimensional case, the groups of Möbius and projective transformations coincide and so in this example the symmetry can of either kind and the concept of a wurf or a cross-ratio, which is an invariant of projective transformations, can be used [2].

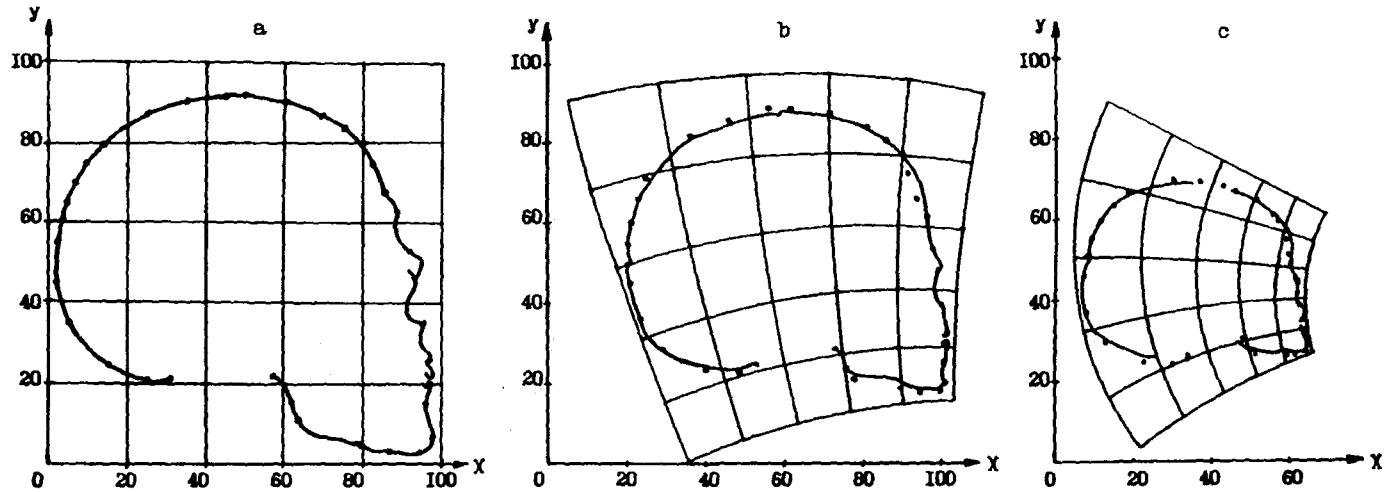

Fig. 22. Möbius transformations in the modeling of ontogenetic transformations of the human skull. Profiles of the skulls of an adult (a), a 5-year-old (b) and a newborn (c), taken from Ref. [32].

Table 1. The wurfs W of the human extremities and torso at different ages

	Shoulder A_1B_1	Forearm B_1C_1	Wrist C_1D_1	W	Hip A_2B_2	Shin B_2C_2	Foot C_2D_2	W	Upper part	Torso	Lower part	W
Embryo age (lunar months)												
4	2.37	1.90	1.55	1.33	2.76	2.66	1.84	1.26				
6	4.40	3.65	3.23	1.34	5.45	5.45	4.30	1.27				
8	6.17	5.05	4.53	1.35	7.40	7.60	5.80	1.27				
9	6.95	5.70	5.14	1.35	8.40	8.60	6.61	1.27				
Newborn	8.80	7.40	6.30	1.33	10.90	9.90	7.80	1.30	10.6	18.3	28.6	1.29
Age (years)												
1	12.5	10.4	8.7	1.33	15.8	14.4	11.5	1.30	15.6	25.0	41.7	1.32
4	16.7	13.7	11.2	1.33	22.4	20.2	15.8	1.30	18.9	31.4	58.4	1.32
7	20.7	16.5	13.1	1.33	28.0	24.8	18.9	1.30	21.0	35.3	71.7	1.33
10	23.2	18.4	14.4	1.32	32.1	27.8	20.7	1.30	22.5	38.0	80.6	1.34
13	26.3	20.7	16.0	1.32	37.7	32.2	23.6	1.30	24.5	42.2	93.5	1.34
17	30.5	23.9	18.3	1.32	44.4	37.3	26.9	1.29	26.9	47.8	108.6	1.33
20	32.3	24.5	18.8	1.33	45.4	37.5	27.0	1.30	25.3	51.8	109.9	1.29
Average and boundaries of ontogenetic fluctuations of W (%)	$1.33 \pm 1.5\% = 1.01P \pm 1.5\%$				$1.29 \pm 2\% = 0.98P \pm 2\%$				$1.32 \pm 2\% = 1.01P \pm 2\%$			

$P = 1.309\ldots$—golden wurf

The lengths (in cm) are taken from the work of Bunak [33].

Table 2. Wurfs W of the middle finger in man at different ages

Age (years)	AB	BC	CD	W
4	2.42	1.43	0.86	1.31
6	2.64	1.65	1.02	1.31
8	3.00	1.88	1.19	1.31
10	3.10	1.96	1.25	1.31
12	3.34	2.13	1.37	1.31
14	3.56	2.27	1.46	1.31
16	4.08	2.57	1.64	1.31
18	4.19	2.65	1.69	1.31
21	4.41	2.78	1.76	1.31

The lengths of phalanxes (in cm) are taken from the work of Rokhlin [34]. AB—the basal phalanx; BC—the medium phalanx; CD—the end phalanx.

V. V. Bunak [33] and D. G. Rokhlin [34]:

$$W = \frac{(C - A)(D - B)}{(C - B)(D - A)}, \tag{3}$$

the expressions in parentheses being the lengths between the end dividing points A, B, C and D of the three members of every block. The values of these wurfs in all the blocks, at least during the entire individual post-natal development, group around the benchmark of $P \approx 1.31$ (see Tables 1 and 2). This is especially interesting because the growth of the human body is essentially nonlinear (Fig. 23); for instance, the upper part grows 2.4-fold, the torso 2.8-fold and the lower part 3.8-fold.

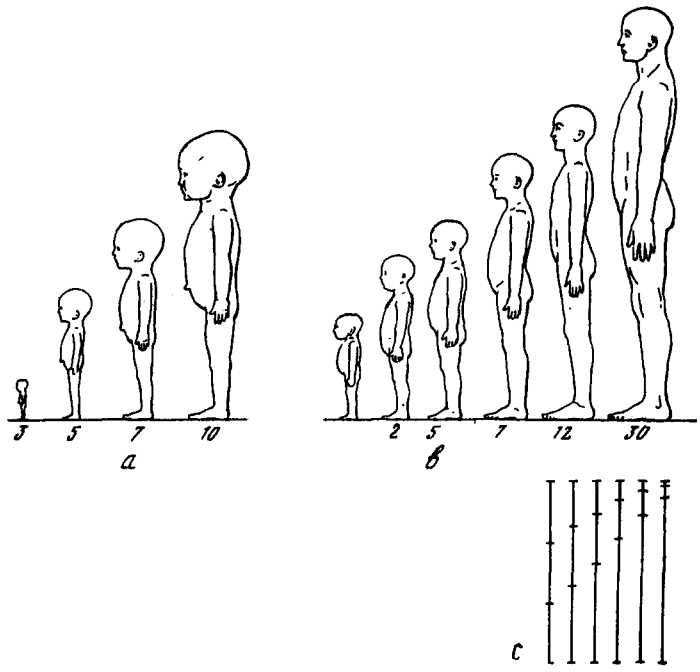

Fig. 23. Changes of the human body with age (according to Ref. [32]). (a, b) Antenatal and postnatal stages—(a) in lunar months (b) in years—the first on the left is a newborn. (c) Three-stretch parts whose wurfs are equal to 1.31.

A non-Euclidean analysis of the structure and growth reveals that all the three-membered blocks of the human kinematics in the straighted posture are practically Möbius equivalent and Möbius invariable during the lifetime. This is largely true of dwarfs and giants and a broad range of highly organized animals as well as normal human subjects. These findings are discussed in more detail elsewhere [35], where (see also the Appendix) it is shown that the value of wurfs, 1.31, in the body blocks is a function of the biological phyllotaxis laws of shaping; it coincides with the so-called golden wurf,

$$P = \frac{3 + \sqrt{5}}{2} = \frac{\Phi^2}{2} = 1.309\ldots,$$

conjugated with the Fibonacci numbers and the golden section,

$$\Phi = \frac{1 + \sqrt{5}}{2} = 1.618\ldots.$$

6. AUTONOMOUS AUTOMATA AND GROUP INVARIANT PROPERTIES OF LIVING ORGANISMS

"The basic feature of every new scientific idea is that it relates in some way two different series of facts." (M. Planck [36])

One of the promising ways to model the phenomena of biological shaping is application of the theory of growing automata to the modeling of morphogeneses [37]. This control engineering modeling is legitimized by the fact that in biological shaping a feedback control system is involved which is distributed throughout the developing body. Cellular automata, uniform arrays devised by von Neumann [38] and his followers, and the ideas of Lindenmayer's parallel grammars [39] are widely used for this purpose. In the light of the data presented in the preceding sections on the important morphological value of iterative algorithms, a sophisticated modeling approach can be developed where the concept of an autonomously growing automaton (or networks of such automata) would be useful.

Recall that a finite automaton is a dynamic system whose behavior at specified times (clock times), 1, 2, . . . , p, is described by the equation

$$x(p) = f[x(p-1), u(p-1)], \tag{4}$$

where $x(p)$ and $u(p)$ are variables which take on values from specified finite alphabets: $x(p)$ representing the internal state of the automaton at time p; $u(p-1)$, the state of the automaton input at the preceding time, which reflects the impact of the "environment" on the automaton. A special case is an automaton whose behavior does not depend on the environment at all. Such an automaton is referred to as autonomous; for such an automaton a change of state is obviously dictated (in a way similar to algorithms of biological cyclomerisms) by the iterative algorithm

$$x(p) = f[x(p-1)]. \tag{5}$$

But biological cyclomerisms are observed in relatively autonomous subsystems of the body such as horns, fins etc. In addition, the concept of autonomy is generally closely related with the genetic inheritance of shapes characteristic of the biological species independently of the environment. The relative autonomy of the subsystems in an organism is a prerequisite of its efficient functioning. All this confirms that our approach is correct in interpreting the biological cyclomerisms and their ensembles in terms of autonomous growing automata [equation (5)]. Other authors used growing automata, assuming timed control of the automaton state from outside while the morphogenetic significance and potential of autonomous automata were neglected. One should not overlook the well-known noise immunity of iterative algorithms which may make them especially desirable in living organisms.

Such modeling can be implemented, for instance, in a cellular automaton—which is usually a uniform array of numerous identical cells, each cell having several possible states and interacting only with a few neighboring cells. The idea of such an automaton is nearly as old as that of electronic computers. The research in this field was pioneered in the early 1950s by von Neumann. The "Life" game, devised in 1970 by J. Convay, and capable of simulating numerous aspects of biological development is the most widely known cellular automaton. Numerous problems in cellular automata are covered by the so-called "information mechanics". What is important is that von Neumann's automata include cells which are placed in the cells of the Cartesian network and this "Euclidean" disposition of cells, introduced from outside, is taken up in later papers. Breaking with this tradition, I used the data on the biological significance of non-Euclidean cyclomerisms to justify the use of cellular automata also based on networks with cyclic groups of non-Euclidean automorphisms in biological modeling. In the "non-Euclidean" cellular automata the timed change of state of the autonomous automaton is treated as attachment of a new motive unit of a non-Euclidean cyclomerism.

The research revealed new promising lines for the development of the theory of growing automata to be applied to biological morphogenesis. Firstly, the specific features of biological morphogenesis make it possible to develop a theory of autonomous automata which are hierarchically embedded into one another and functioning at every level with different clock time periods which are or are not multiples of those of the preceding or succeeding levels. The case of

nonmultiple periods fits well the biological situation of forming, for instance, leave organs on a cylcomeric plant stalk, or structures which develop at a later stage relatively autonomously by their shaping algorithms.

Secondly, the idea of a succession of autonomy conditions f in equation (5) proves useful in interpreting the series of biological data on the dependence of the shape of cylcomeric structures on the specifics of the environment. Indeed, in a great number of cases a change of the habitation range for a species entails a change in their cyclomerisms, in that the kind of cyclomeric change which is easily interpretable in terms of changes of the autonomy conditions. This approach is equally applicable to other phenomena of cyclomeric polymorphism.

Thirdly, in addition to autonomous automata, the concept of a quasiautonomous automaton is useful in analysis of the body of morphological data, such an automaton is described by equation (4) with $u(p)$ making either little impact or only changing the generating transformation coefficients f rather than shape. In addition, variations of $u(p)$ with clock times may be stochastic as well as deterministic. This approach easily leads to the description of biological structures whose parts are interrelated in a random and irregular way. The resultant data on the biological significance of iterative algorithms force the writer to a conviction that for mathematical modeling of a broad range of biological development phenomena formalisms of discrete (or finits) mathematics are more adequate.

Furthermore, the idea arises that numerous organism subsystems function in the norm and in pathology as autonomous automata and their ensembles may prove important in medicine and biology and suggest treatment methods, professional selection and training procedures, ways to optimize the working conditions for operators of complex machinery etc.

The laws of interaction between parts in biostructures may be related with the concept of the biological space. In formal usage of physical terminology, the world of a biological body is a manifold of events where a system exists of events making an impact on other events which dictate its space structure. In that world the physical understanding of space and space–time is applicable, formulated by A. D. Aleksandrov (Member of the U.S.S.R. Academy of Science) in 1965,

> "Space–time is a set of various events in the world taken apart from all its properties other than those that are determined by the structure of the system of relations made by impacts of events on one another." [40, p. 55]

In our view, the whole issue of general interaction laws in the biological space must be treated as laws of interaction in the spaces of self-organizing ensembles of self-organizing growing automata. The geometrization of biology seems to be contingent on problem-oriented development of the theory of finite automata. In light of the preceding discussion, the following definition may prove useful: a biological space is space of self-organizing hierarchical ensembles of self-organizing growing aperiodic (in the general case) automata operating with different clock time periods. For brevity this ensemble may be referred to as a hyperautomaton.

In this formulation aperiodic implies rhythmically developing, in time, biological processes whose time structuralization may prove to be controlled by nontrivial iterative algorithms which will be treated in the following section.

7. APERIODIC ITERATIVE BIORHYTHMS

> "Whenever you have to deal with a structure endowed entity E try to determine its group of automorphisms, you can expect to gain a deep insight into the constitution of E in this way. (H. Weyl [3, p. 158])

Researchers in various countries have, for a long time, been studying time biorhythms and this field of natural sciences can claim its own traditions, terminology and challenging findings. Still, it has concentrated attention on periodic rhythms of physiological processes such as breathing and walking, repeating processes occurring simultaneously with periodic diurnal and seasonal changes etc. The reader of the literature on biorhythms may think that no biorhythms other than periodic are significant or possible. In point of fact, however, the range of biologically significant rhythms is broader and the periodic ones is a particular, albeit important, case.

Fig. 24. The evacuation rhythms in *Arenicola marina* [42] described as time cyclomeres with a Möbius generating transformation $t_{k+1} = (1.292\, t_k + 0.593):(0.020\, t_k + 1)$. The arrow indicates the time of high tide when the observation terminates.

The periodic rhythms of any process may be interpreted as structuring by a transitive iterative algorithm (1) whose generating transformation is a parallel shift along the time axis by the period.

But are less trivial iterative algorithms, whose time structuring is dictated by more complicated generating transformations, not implemented in biological processes? This very important question, which is answered in the positive, draws attention to specific biorhythms and regularities in time structuring that are usually neglected but seem to be worthy of most serious attention and capable of significantly enriching the science of biological rhythms.

An illustrative example of an iterative algorithm whose generating transformation in scale similarity is the well-known moulting of crustacea [41]. The time between two moults is seen to increase monotonically with a constant scale during the entire lifetime over which moults occur. Not only is the total time between moults scaled but also, and with the same factors, the duration of every stage in the preparation for a moult. In addition, every moult is followed by a proportional increase in the size and mass of the organism. In other words, this biological transformation is a case of a remarkable organization of the most complicated biological processes in space–time which demonstrates the cyclomeric properties and adds legitimacy to the search for symmetrical structuring algorithms in time biorhythms that would be similar to such algorithms in biological series in space.

Aperiodic iterative algorithms with a Möbius (or projective, which is the same in a unidimensional case) generating transformation

$$t_{k+1} = \frac{1.292 t_k + 0.593}{-0.02 t_k + 1} \tag{6}$$

are demonstrated by, for instance, the *Arenicola marina's* rhythmic evacuation (Fig. 24), which follows from our analysis of experimental data reported by P. R. Evans [42]. In laboratories these worm-like organisms demonstrate an automatic periodic rhythm of excretion with an interval of about 40 min. In natural conditions these creatures observe a quite different rhythm in which the intervals grow nonlinearly and follow algorithm (6). These organisms were observed in their natural milieu at low tide. The moment they are engulfed by high tide is shown in Fig. 24 by a vertical arrow.

Nontrivial Euclidean and non-Euclidean iterative algorithms are obviously at work in other rhythmic processes as well, such as the volley-like flapping of an excited *Bonasa umbellus* heath cock, some kinds of heart arhythmias etc. The analysis of non-Euclidean symmetries in biological processes such as complex structured communication signals between living organisms, rhythmical change of organism state parameters in normal and pathological functioning etc. must undoubtedly be continued and systematized. The search for such symmetries in morphogenetic processes such as moulting is especially important and challenging for general biology. The modeling of such processes could rely on the theory of aperiodic automata.

8. THE PROTOSYSTEM OF BIOLOGICAL REGULATION AND PSYCHOPHYSICAL MATTERS

"To take possession of space is the first gesture of the living, men and beasts, plants and clouds, the fundamental manifestation of equilibrium and permanence. The first proof of existence is to occupy space." (Le Corbusier [43, p. 30])

"I'm looking with tactile eyes. I'm feeling with a seeing hand." (J. W. Goethe)

The recognition of non-Euclidean symmetries in algorithmical mutual conjugation of parts of organic forms has dramatically extended the range of biological objects, the spatial behavior of whose parts may be quite legitimately and stringently described in geometrical terms. The existence

in most diverse biological bodies of the same geometrical types of cooperative structure and behavior is seen as a major feature of biological evolution and suggests the existence and general biological significance of a morphogenetic regulation system which is responsible for this coordinated behavior of body parts in the growth and development of the organism.

This morphogenetic system of integrating regulation seems to exist concurrently with the nervous and humoral systems which also perform certain functions in integrating parts of the organism. The system seems to be a protosystem, above all in that it emerged earlier in the evolution (for the ability to grow is the most ancient property of living matter; morphogenesis and growth of geometrically regular and nontrivial forms is observed in protozoa, in particular, in unicellular organisms which have no nervous system). In other words, other systems are immersed in this one by their structural formations and also emerge in philo- and ontogenesis against its background.

Even in highly sophisticated organisms one can see at certain stages the secondary nature of incorporating the nervous system into the regulation of the behavior of organs and tissues in the course of individual development; before the essential elements of the nervous system emerge, parts of the organism act vigorously in a coordinated way similar to morphogenetic movements. For illustration let us recall two well-known biological facts [44]. (1) The first organ to move in a 4 mm long 3-week-old fetus is the heart; it starts beating for internal reasons because at that stage the heart has no nervous links; the heart contracts when there is no blood to pump. (2) The larvae of the *Tantogolabrus* fish on the first day of free swimming in a body of water do not respond to exogenous irritants or move without an effectual receptor system. The sensory system develops gradually in them to master the primary motion system for biologically efficient coordination of its in-born activity with the nature of exogenous irritation.

Note also that the three-dimensional growth, which is heavily dependent on the cyclomeric (or cyclogenetic) properties of the living matter, integrates the numerous parts of the body into a growing ensemble which acts to prevent the body transformations which occur with the aging from invalidating all the senso-motoric habits that were acquired at preceding ontogenesis stages when the body was "different". In this context the geometrical properties of three-dimensional growth and the biochemical media which make this growth possible have, in my view, a direct bearing on the formation in man of an inborn idea of the structure of his body, an idea which has for a long time been regarded as a major element in the spatial perception and coordination of movements. The morphogenetic regulation system which largely functions by cyclogenesis performs, among other things, the function of coordinated adjustment of numerous muscular, joint and other proprioceptors which contribute to man's awareness of the structure of his body. The morphogenetic regulation system which looks after three-dimensional growth acts for diverse proprioceptors and muscles as a distributed cooperation enforcing unit, a tuning fork. In other words, the various elements of the senso-motoric system act in unison, not only by virtue of their direct mutual links but also because they are immersed in an orderly growing environment which influences their operation and aids the brain in composing generalizing adequate images out of innumerable reports from the set of receptors. Without studying the morphogenetic regulation protosystem and its general biological significance one cannot understand to the full the more recent organism regulation systems, in particular the nervous system.

Now let us consider non-Euclidean symmetries in psychophysical phenomena. The idea of a close linkage between the specifics of spatial perception and the morphogenesis principles is deeply rooted. Since the times of Kepler the esthetic quality of the golden section has been related in the literature with its morphogenetic materialization in living bodies. Today's psychology has had much experience in using the higher geometry in similating the observed psychological phenomena.

Back in 1868, H. Helmholtz [45] tried to apply a certain mathematical concept related with the concept of groups of transformations to the psychology of perception. The desire to obtain an insight into the psychological phenomena in perception from the basic viewpoint of the concept of a group led H. Poincaré [46] to an interesting concept on the genesis of spatial ideas in an individual, and on the interrelationships between physical, logical and physiological fundamentals of geometry. Poincaré developed and consolidated this idea until he died. He believed that a group should be accepted as a key concept in the scientific cognition of the genesis of spatial ideas in an individual. He thought that the concept of a group, at least potentially, was in-born and existed before any individual experience.

With time the mathematical concept of a group won new ground in psychological research, notably in the so-called perception constance phenomena. In his enlightening paper, E. Cassirer [5] indicated that the analogy between the psychology of perception and geometry is, above all, in that for the perceived patterns (shapes) the specific properties are preserved even when their elements change. Thus, a musical tune does not change much if all its notes are shifted. This is also true of visual perception by dimensions, shape, color etc.

> "Indeed, what else is that 'identity' of the perceptual form but what, in a much higher degree of precision, we found to subsist in the domain of geometrical concepts? What we find in both cases are invariances with respect to variations undergone by the primitive elements out of which a form is constructed. The peculiar kind of 'identity' that is attributed to apparently altogether heterogeneous figures by virtue of their being transformable into one another by means of certain operations defining a group, is thus seen to exist also in the domain of perception." [5, p. 25].

Klein's Erlanger program suggests essentially that not every interpretation of a three-dimensional object is its geometrical characteristic. If we take an object and recognize only its individuality, its geometrical nature and significance would not manifest itself. According to Cassirer [5], a description of a spatial shape as such individually yields, at best, its geographical, or "topographical" rather than geometrical characteristic. To obtain the latter, a quite different approach is needed which was formulated by Klein as follows: the geometrical properties of any figures must be described by formulas which remain true when the system of coordinates changes; conversely, any formula which is in this sense invariant with the group of available transformations of coordinates represents a geometrical property. In compliance with this concept the properties which describe an object should not be defined for elements which add up to this object. This should only be done in the framework of the group with which the object is related. As soon as a group is replaced, quite new relationships emerge. What expressed "the same" geometrical concept may obtain a significance of its own whereas what appeared to be something quite different may prove to be identical in the new geometry. Thus, in terms of the metrical Euclidean geometry various conic sections (circumference, ellips, parabola and hyperbola) act as independent geometrical patterns having their own specific and distinct properties. This difference disappears when the same conic sections are regarded in terms of projective geometry because projective transformations may translate all the above conic sections into one another and thus deprive them of independent existence. In effect, the concepts of today's geometry acquire their inherent accuracy and become truly universal only insofar as the specific figures, suggested by intuition, are not regarded as predestined or rigid but rather as raw material which needs processing if it is to acquire a geometrical status. The mathematical definitiveness is not in the elements which the mathematicians treat as given quantities but in the rule whereby these elements are interrelated. Ever since the Erlanger program, nobody has asked whether two quadrangles are identical or whether two stretches are equally long without reference to a specific group of transformations within which the question must be answered. In other words, the definitiveness of a figure depends on the context where it is integrated and whose particular case it is.

Bertrand Russell who worked on geometrization of psychology wrote:

> "Il faut construire un pont en commencant à la fois par ses deux extrémités: c'est-à-dire, d'une part, en rapprochant les assomptions de la physique des données psychologiques et, de l'autre, en manipulant us données psychologiques de manieur à édifier des constructions logiques satisfaisant de plus prés aux axiomes de la géométrie physique." [47, p. VIII].

Are symmetrical principles of biomorphogenesis not used in the psychophysics of spatial perception? Does the organism not tend to perceive the world in terms of patterns of the same groups of transformations in compliance with whose principles it is shaped? Are the structuring principles that materialize in morphogenetic phenomena and psychological phenomena of spatial perception not akin?

These questions are especially natural because Helmholtz, Sechenov and Poincaré explored the idea of the leading role of the kinetic structure of the body (which is now known to be structured largely by morphogenetic principles with Möbius and other non-Euclidean symmetries) in the genesis of spatial concepts in an individual. In particular, this idea was the cornerstone of Poincaré's teaching on the physiological fundamentals of geometry and on the origin of spatial ideas in an individual [46]. (At this point it is important to remind the reader that the argument

advanced in this paper on the essential importance of non-Euclidean geometry for this range of biological issues was not to be found in the writings of those authors.)

According to Poincaré, the very notion of space and geometry originates in an individual because of the activities of the body kinematics which incorporates the internal receptors of the attitudes and movements of body parts with respect to one another; i.e. there is in the kinematic arrangement something which precedes the notion of space. The specifics of the entire apparatus behind the kinematic activities of the body has been made consistent with the realities of the world by evolution. Consequently, every newborn organism masters adequate spatial notions not only through personal contact in the course of ontogenesis with objects of the environment, but also thanks to what preceding generations acquired and was consolidated in the course of phylogenesis in the body movement apparatus.

Poincaré believed that the body movements had a leading role to play in the genesis of the space concept. For an immobile being there would have been no space or geometry. If we had no measuring tool, Poincaré insisted, we could not build space; but we have and we use it instinctively—it is our own body.

> "The system of coordinate axes to which we refer all external objects is a system of axes invariably references to our body which we carry wherever we go." [46]

However, Poincaré and numerous other authors did not practically ask an important question, is it Euclidean or non-Euclidean groups of transformations and the related reference systems that are characteristics of the movement and space perception in man and animals? What we argue is that groups of non-Euclidean transformations and cyclogenetic principles contribute to the genesis of space ideas in the individual. The following discussion will advance more evidence in favor of this argument.

As far as Poincaré's ideas on a human body reference system of coordinates are concerned, all the elements of the human body change their Euclidean lengths as they grow and at the fastest rate in infancy when the organism finds its way in the world and when stable references of length are especially important. Although being deprived of a reference of the Euclidean measure, the body has invariably during its lifetime and in the life of the entire species references of non-Euclidean wurfs (Section 5), which are quite good as tools for geometric comparison and construction of coordinates.

Speaking of Möbius biosymmetries of kinematic blocks in the human body and the Poincaré theory, we have to recall the date of today's physiology on kinesthesia which are found to be closely related with those separating points whose positioning in the body is found to be Möbius. The kinesthesia mechanisms supply knowledge on the attitude and the movement of body elements, Poincaré believed that these ideas stemmed from the "muscular sensation". A modern physiological handbook states, however, that the sense of position and of movements of the joints is provided solely by appropriate receptors in the joints and there is no need in the enigmatic "muscular" sensation to explain the kinesthetic sensations [48].

Today's physiology also supplements Poincaré's teaching on the interrelationship of the body and spatial sensations by asserting that the ideas the individual has on his bodily shape are in-born. This assertion stems from studies of the so-called phantom sensations in invalids who feel the presence of a lost part of the body, not only in those whose extremities were amputated but also in those who were born without extremities. Consequently, the individual's idea of the structure of his body is in-born rather than dictated by his experience. This seems to be another argument in favor of the morphogenetic mechanisms contributing to the formation of spatial feelings in an individual.

The significance of the morphogenetic structures for spatial perception and active ranking of the environment by the organism is demonstrated by constructions made by insects and other organisms. Usually instinctive and following certain standards of shape, this activity is an enigma and a scientific problem. In numerous cases these construction works of individual organisms and joint efforts of numerous individuals appear to embody Euclidean cyclomeres. Zoopsychology (the stigmetry theory) knows that the construction shape is an active factor in instinctive construction activities and manages the work by itself.

The morphofunctional structure of the sensory systems in an organism embodies nontrivial

morphogenetic symmetries and algorithms. Thus the human cochlea consists of three patterns, the ratios of whose lengths make the golden section Φ ($I:\Phi:\Phi^2$) and yield the golden wurf, 1.31 (Fig. 25), of Section 5, where other morphogenetic shapes, unrelated to the perception of sound were discussed. The cochlea shape is usually represented schematically as a logarithmic helix (which reproduces the classical similarity cyclomere) inside which lies Corti's organ of a kindred shape. In some animals such as cats, however, Corti's organ has a shape which is significantly different at the broad end of the cochlea from the logarithmic helix and tends to curl lyre-like in the inverse direction, and is described on the whole as a loxodromic Möbius cyclomerism.

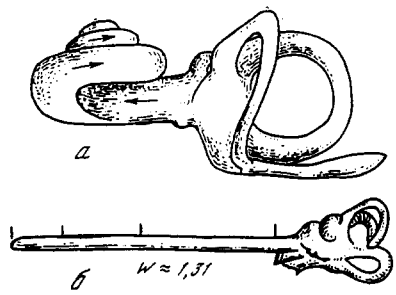

Fig. 25. The helical structure of the human ear cochlea and the golden wurf in it [49]. (a) Cochlea, coiled into a helix; (b) cochlea uncoiled into a straight line.

The non-Euclidean geometry is also applied to the description of visual perception. This kind of research was pioneered by R. Luneburg [50] who proved that the space of visual perception is described by the Lobachevsky geometry. These findings were followed by scores of papers in various countries where the idea of a non-Euclidean space of visual perception was extended and refined. A book by B. V. Rauschenbach [51], a Soviet researcher, is worthy of special attention. The Luneburg approach was thoroughly tested by G. Kienle [52] who arrived at essential conclusions on the importance of the Möbius geometry in structuring the space of visual perception.

In the main series of his experiments, where about 200 observers were involved, Kienle obtained about 1300 visual patterns of various kinds. In addition, 11 additional series yielded over 2000 measurements. The observers made judgments on the straightness of curves; the equality of angles, distances, segments and areas; the parallelism of straight lines; the existence of a common center of a family of circumferences etc. The experiments confirmed that the space of visual perception is described by the Lobachevsky (or hyperbolic) geometry; Kienle came to the conclusion that the well-known conformal (or Möbius) model developed by Poincaré was an adequate model of that geometry. He concluded his paper by writing:

> "Poincaré's model of hyperbolic space, applied for the first time for a mapping of the visual space, shows a reasonably good agreement with experimental results." [52, p. 400].

Consequently, Möbius transformations, more specifically a subgroup which is associated with the Lobachevsky geometry, prove to be the prime factor in structuring the space of visual perception. Why it is that this particular group is so important remains an open question. Other non-Euclidean geometries may also be important for other kinds of spatial perception and for visual perception under conditions different from those used by Luneburg and Kienle.

In this light it is especially enlightening that Möbius symmetries and iterative algorithms are also to be found in microscopic movements of the eye apple, without which normal visual perception is known to be impossible. Ref. [53, p. 195] reports the path of a point moving along the eye retina against a background of microscopic apple tremor with respect to the retina receptors. The diameter of the central retina area is a mere 0.05 mm, the tremor frequency being up to 150 cycles/s and the amplitude approximately equal to half the diameter of a conecell. The nature of the image tremor is regular in that all five wave-shaped retina image drift lines with tremor show either 27 or 42 saw-like periods. Every tooth of the saw-like path is commensurable with neighboring salients and approximately orthogonal to the curvilinear draft path; in other words, the saw-like periods

are integrated into a regular process. Our research into the structure of these curvilinear paths with a high frequency tremor has demonstrated that they are satisfactorily described as Möbius cyclomeres. The iterative algorithmical shape of the microtremor may be thought to be dictated by some autonomous automaton controlling the eye tremor.

Sensor perception, as in the above cases of biological kinematics, is largely structurally related with morphogenesis and may be treated as a structural extension of morphogenesis rather than something entirely new that nature devised at some evolutionary stage. In "mastering space" the organism seems to use the same structuring principles and algorithms, be it morphogenetic, kinematic or psychophysical mastering. This unity of structuring principles and algorithms for various systems and sophistication levels being coordinated and adding up to a living organism which "masters space" offers important advantages to living organisms. The writer is confident that the biological significance of iterative algorithms and non-Euclidean symmetry will be detected in the most diverse biological fields, in particular, in the encoding, notably in the nervous system, of biological information.

9. CONCLUSIONS

For biology and the mathematical modeling of biological phenomena to advance, the symmetrical and algorithmical properties of organic shaping need in-depth study. This article sheds light on previously unknown basic properties of morphological self-organization of living matter, observable in a broad range of supramolecular organic bodies of different evolutionary classes. These properties manifest themselves in that the structure and behavior of the set of individual parts in algorithmically organized biological structures are coordinated by finite parametrical groups (and semi-groups) of non-Euclidean transformations, above all Möbius and affine.

Shubnikov and Koptsik [54, p. 26] define symmetry as the broadest maximal group of object automorphisms. The author's own finding is that by disregarding non-Euclidean automorphisms biology overlooks symmetries and the very fact that quite a few living beings are symmetrically algorithmically structured. Bearing this in mind, morphology would have to dramatically expand its fundamentals by using non-Euclidean, rather than the conventional similarity, transformations if group invariant tools of modeling are to be efficiently applied and refined.

The discovery of non-Euclidean biological symmetries provides valuable data for the development of a formal theory of morphogenesis, the ability to explain the existence of these symmetries being an illustrative indicator of the adequacy of this theory. A reliable method has been found for the materialization of Vernadsky's idea [10] regarding adequate employment of the non-Euclidean geometry in biomorphology along the lines of the Erlanger program and one which makes it possible to develop biological theories in the spirit of the "geometrization of physics".

A recent paper [55] on nonbiological applications of iterative algorithms (unrelated with Euclidean, Möbius or projective transformations) describes a number of biological-like structures and notes that "these curious configurations demonstrated that the rules whereby the most complicated living tissues are structured may be plain to the point of absurdity".

The findings reported in this article go far beyond those results as far as relations with conventional biomorphology, non-Euclidean geometries, theory of automata, the kinematics of biological movements etc. are concerned.

Research in biosymmetries enhances the comprehension of the unity of nature, in the same vein that science discovers ever new general biological laws and mechanisms such as the genetic codes and bioenergy mechanisms. This article sheds light on and analyzes new, non-Euclidean, symmetrical and algorithmical properties of general biological phenomena in biological morphogenesis, the knowledge of which is indispensible for a broad range of theoretical and applied areas, including anthropological and zoomorphological robotics, ergonomics, biomechanics, biotechnology (above all, "morphoengineering", or directed morphogenesis control) etc. These results emphasize the morphological significance of the internal environment of the organism and suggest new approaches to understanding their relationship of morphogenesis and the mechanisms of genetic encoding and biochemical cycles [56]. The writer's findings are not only in favor of the argument that biology is a fertile field for the introduction of various symmetrical approaches, methods and tools of group-theoretical analysis [3, 54, 57 *inter alia*], but that the development of

theoretical biology and biomechamics at this stage is largely dependent on vigorous utilization of group-theoretical methods with the use of non-Euclidean geometries.

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REFERENCES

1. V. P. Vizgin, *The Erlanger Program and Physics* (in Russian). Nauka, Moscow (1975).
2. F. Klein, *Vorlesungen über nicht-Euklidische Geometrie*. Springer, Berlin (1928).
3. H. Weyl, *Symmetry*. Princeton Univ. Press, N.J. (1952).
4. P. Descartes, *Selected Works* (in Russian). Izdatelstvo inostrannoy literatury, Moscow (1959).
5. E. Cassirer, The concept of group and the theory of perception. *Phil. phenomenol. Res.* **5**(1), 1–36 (1944).
6. J. R. Derome, Biological similarity and group theory. *J. theor. Biol.* **65**(2), 366–378 (1977).
7. S. V. Petukhov, Studies in non-Euclidean biomechanics. In *The Biomechanics of Man–Machine Systems* (in Russian), pp. 38–39. Nauka, Moscow (1981).
8. A. Eddington, *The Philosophy of Physical Science*. Cambridge Univ. Press, Cambs. (1939).
9. R. Thom, *Structural Stability and Morphogenesis, an Outline of a General Theory of Model*. Benjamin, Boston, Mass. (1975).
10. V. I. Vernadsky, *The Chemical Structure of the Biosphere and its Environment* (in Russian). Nauka, Moscow (1965).
11. S. V. Meyen, B. S. Sokolov and Y. A. Shreider, The classical and non-classical biology: the Lyubishchev phenomenon. *Vest. Akad. Nauk SSSR* **10**, 112–124 (1977).
12. Symmetry and function of biological systems at the macromolecular level. In *Proc. 11th Nobel Symp.*, Stockholm (1969).
13. A. V. Shubnikov, The similarity symmetry. *Kristallografiya* **5**(4), 489–496 (1960).
14. S. Lie, *Theorie der Transformationsgruppen*, Ab. 3. Verlag, Leipzig (1983).
15. C. Reid, *Hilbert*. Springer, New York (1970).
16. L. A. Zenkevich (Chief Ed.), *The Animal Life* (in Russian). Prosveshcheniye, Moscow (1976).
17. E. Haeckel, *Naturlicheschopfungs Geschichte*. Verlag, Berlin (1898).
18. Meyen S. V., Plant morphology in its nomogenetical aspects. *Bot. Rev.* **39**(3), 205–260.
19. E. Haeckel, *Kunstformen der Natur*. Verlag, Leipzig (1904).
20. Al. A. Fyodorov (Chief Ed.), *The Vegetation Life* (in Russian). Prosveshcheniye, Moscow (1976).
21. M. G. Pertushevskaya, *Radiolaries of the Nassellaria order in the World's Oceans* (in Russian). Nauka, Leningrad (1981).
22. J. Bernal and S. Carlyle, Ranges of generalized crystallography. *Kristallografiya* **13**(5), 927–951 (1968).
23. Yu. A. Urmantsev, The specifics of space and time relations in living nature. In *Space, Time, Movement* (in Russian), pp. 215–241. Nauka, Moscow (1971).
24. D'Arcy W. Thompson, *On Growth and Form*, 2nd edn. Cambridge Univ. Press, New York (1942).
25. D. V. Nalivkin, Curvilinear symmetry. In *Crystallography* (in Russian), pp. 15–23. Metalurgizdat, Moscow (1951).
26. N. A. Bernstein, *Essays in the Physiology of Movements and the Physiology of Activity* (in Russian). Meditsina, Moscow (1966).
27. O. M. Ivanova-Kazas, *Comparative Embryology of Invertebrates* (in Russian). Nauka, Novosibirsk (1975).
28. L. S. Berg, *The USSR Food—Fish* (in Russian). Pishchepromizdat, Moscow (1949).
29. L. S. Berg, *Selected Works* (in Russian). Nauka, Moscow (1961).
30. A. Weiss, Replication and evolution in inorganic systems. *Angew. Chem.* **20**(10), 850–860 (1981).
31. W. W. Sawyer, *Prelude to Mathematics*. Penguin Books, Harmondsworth, Middx. (1955).
32. B. M. Petten, *The Human Embryology* (in Russian). Medgiz, Moscow (1959).
33. V. V. Bunak, Changes in the relative length of human extremity skeleton segments during the growth period. *Izv. Akad. Nauk RSFSR* **84**, 33–45 (1957).
34. D. G. Rokhlin, *X-ray Osteology and X-ray Anthropology* (in Russian). Biommedgiz, Moscow (1936).
35. S. V. Petukhov, *Biomechanics, Bionics and Symmetry* (in Russian). Nauka, Moscow (1981).
36. M. Planck, *Vortrage und Errinnerungen. Ursprung und Auswirkung Wissenschaftlichen Ideen*. Springer, Stuttgart (1949).
37. M. J. Apter, *Cybernetics and Development*. Pergamon Press, New York (1967).
38. J. von Neumann, *Theory of Re-producing Automata*. Univ. of Illinois Press, Urbana (1966).
39. A. Lindenmayer, Algorithms for plant morphogenesis. In *Theoretical Plant Morphology* (Ed. R. Sattler. Leiden Univ. Press, London (1978).
40. E. N. Andreyev, *The Microworld Space* (in Russia). Nauka, Moscow (1969).
41. T. H. Waterman (Ed.), *The Physiology of Crustacea*. Academic Press, New York (1960).
42. P. R. Evans, Adaptations shown by foraging shorebirds to cyclical variations in the activity and availability of their intertidal invertebrate prey. In *Cyclic Phenomena in Marine Plants and Animals* (Eds E. Maylor and R. Hartnoll), pp. 357–366. Pergamon Press, New York (1979).
43. Sh.-E. Le Corbusier, *The Modular*. Faber & Faber, London (1963).
44. J. M. R. Delgado, *Physical Control of the Mind, Toward a Psychocivilized Society*. Harper & Row, New York (1969).
45. H. Helmholtz, *Über die Tatsachen, die der Geometrie zu Grunde liegen*. Teubher, Berlin (1868).
46. H. Poincaré, *Oeuvres*, Vol. II. Gauthier-Villars, Paris (1916).
47. J. Nicod, *La Geometrie dans le Monde Sensible* (Pref. de B. Russell). Press Univ. France, Paris (1962).
48. J. E. Rose and V. B. Mountcastle, Touch and kinesthesis. In *Handbook of Physiology*, pp. 387–429. Am. Physiol. Soc., Washington, D.C. (1959).
49. T. A. Cook, *The Curves of Life*. Constable, London (1914).
50. R. K. Luneburg, The metric of binocular visual space. *J. opt. Soc. Am.* **40**(10), 627–642 (1950).
51. B. V. Rauschenbach, *Spatial Designs in Old Russian Paintings* (in Russian). Nauka, Moscow (1975).
52. G. Kienle, Experiments concerning the non-Euclidean structure of visual space. In *Bioastronautics*, pp. 386–400. Pergamon Press, New York (1964).

53. *Perception; Mechanisms and Models*. Freeman, San Francisco, Calif. (1972).
54. A. V. Shubnikov and V. A. Koptsik, *Symmetry in Science and Art* (in Russian). Nauka, Moscow (1972).
55. A. K. Dudeney, Wallpaper pattern generating programs. *V mire nauk*. **11**, 92–99 (1986).
56. L. V. Belousov, *The Biological Morphogenesis* (in Russian). Moscow Univ. Press, Moscow (1987).
57. B. K. Vainshtein, *Today's Crystallography: Symmetry of Crystals, Structural Crystallography Methods* (in Russian). Nauka, Moscow (1979).
58. S. P. Kurdyumov, Combustion eigenfunctions in a nonlinear environment and its structural laws. In *Contemporary Issues in Mathematical Physics and Computing Mathematics* (in Russian), pp. 217–243. Nauka, Moscow (1982).

APPENDIX

Mathematical Formalisms

Let us recall what the basic groups of transformations in this article, Fibonacci numbers, and iterative algorithms in various applied mathematical fields are.

For a three-dimensional space, the following interrelationship of finite groups of pointwise transformations hold, formulated by Lie back in 1893 [14]:

$$D \subset S \subset \begin{cases} M \\ LH \subset A \subset Pr, \end{cases}$$

where D is a 6-parametric group of Euclidean movements, S is a 7-parametric similarity group, M is a 10-parameter group of Möbius transformations, LH is an 11-parameter special linear inhomogeneous group, A is a 12-parameter group of affine transformations and Pr is a 15-parameter general projective group (the structure is very much like this one in the case of a two-dimensional space). Every group is associated with a certain geometry: S, Euclidean; M, conformal; Pr, projective. In effect, there are two possible ways to look for generalizations of similarity biosymmetries, conformal geometrical (M) and projective geometrical ($LH \subset A \subset Pr$). Both are described in the main text.

The group of similarity transformations which preserves the shape of figures includes transformations of parallel shift, rotation, mirror reflection and scaling. In a three-dimensional space, the general analytical expression of the transformation has the form

$$x'^i = K \cdot L^i_j x^j + b^i \tag{A.1}$$

(with an orthogonal matrix L^i_j, K being an arbitrary number and $i, j = 1, 2, 3$). The group of projective transformations consists of transformations which have, in Cartesian coordinates, the form

$$x'_i = \frac{c_{ij} x_j + b_i}{d_j x_j + 1}, \tag{A.2}$$

where c_{ij}, b_i and d_j are real coefficients which satisfy the condition that the determinant of the system (A.2) is nonzero, $i, j = 1, 2, 3$. These transformations change linear patterns into nonlinear ones and preserve the membership relations between them (Fig. A1).

The Möbius group has been used in mathematics and physics under different means, e.g. conformal, circular, analagmatic, inverse radii, Möbius, Liouville and Kelvin (after scientists who made significant contributions to studies of its mathematical

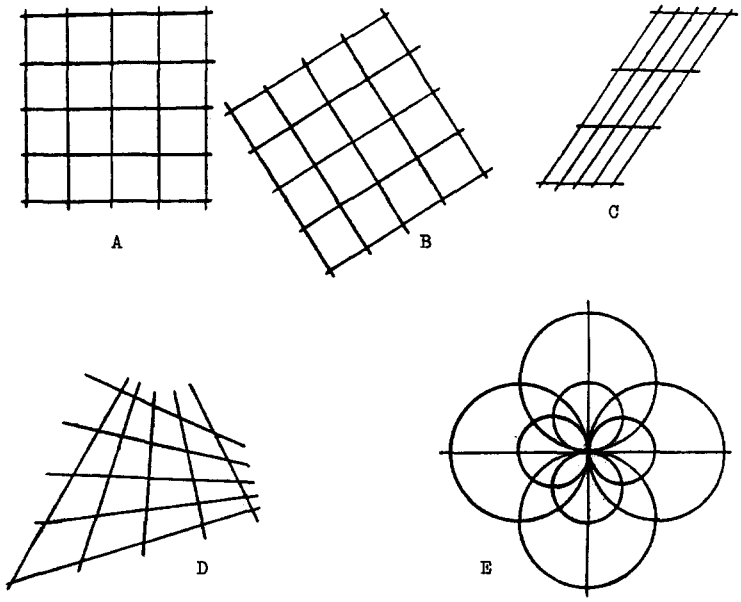

Fig. A1. Characteristic transformation of the Cartesian net (A) by various group transformations: Euclidean (B), affine (C), projective (D) and Möbius (E).

properties and the development of their physical applications). By definition, this is a group of pointwise transformations which preserve the values of angles and transform spheres into spheres. Möbius transformations are locally similar transformations, in that they maintain the shape "in the small" (by the Liouville theorem in the case of a three-dimensional space their group coincides with the group of conformal transformations; in the two-dimensional case the latter group is broader than the Möbius group). This implies that if a small vicinity about every point of a three-dimensional body is scaled, the vicinities about different points not necessarily being scaled equally, the entire body undergoes a stringently Möbius transformation (if the vicinities about all points are scaled to the same extent, the entire body undergoes a similarity transformation which is a particular case of the Möbius transformation). Möbius, or conformal transformations are distinguished from the other groups of transformations of a three-dimensional space by these properties. The local properties of conformal geometrical objects is a major field of differential geometry. Möbius transformations of three-dimensional patterns have a specific set of invariants, including surface curvature lines and their isogonals, ombilical points etc. These transformations divide the set of figures into closed classes of Möbius-equivalent figures that are somewhat broader (because of the additional three parameters of the group) than the conventional classes of equivalent figures in the similarity group. The totality of all Möbius transformations of a three-dimensional space add up to a 10-parameter group, whose analytical expression in canonical pentaspherical coordinates coincides with the group of Lorentz's transformations of 5 variables. Any Möbius transformation in an n-dimensional space may be represented as a superposition of one similarity transformation (A.1) and one inversion with respect to a sphere of unit radius:

$$x_i' = R^2 \frac{x_i - x_{0i}}{\Sigma(x_j - x_{0j})^2} + x_{0i}, \tag{A.3}$$

where R is the inversion (in this case unit) radius, x_{0i} are coordinates of the inversion center and $i, j = 1, 2, \ldots, n$. Any Möbius transformation in an n-dimensional space can also be represented as a product of no more than $n + 2$ common inversion transformations with respect to a sphere. With an infinite inversion radius, inversion with respect to a sphere changes into mirror reflection with respect to a plane.

In the infinitesimal form, Möbius transformations are

$$f^i = a^i + w^{ij}x^i + ax^i + v^j x^j x^i - \tfrac{1}{2}v^i x^j x^i,$$
$$L(x) = 2(a + v^j x^j),$$

where the parameters a^i determine infinitesimal shift transformations; w^{ij}, turn transformations; a, homothetic transformations; and v^i, Möbius proper transformations. Proceeding from this infinitesimal transformation f^i, a finite Möbius transformation is obtained by using Lie equations for determination of finite Lie transformations from given infinitesimal transformations:

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = f^i(x),$$

where t is a group parameter, $i = 1, 2, 3$. Solutions of these equations in this case are a similarity shift transformation $x'^i = x^i + A^i$, a turn transformation $x'^i = 0^{ij}x^j$, $0^{ij}0^{jl} = \delta^{il}$, a homothetic transformation $x'^i = Ax^i$ and a Möbius proper inversion transformation (A.3).

Now let us consider the series of Fibonacci numbers, which is a recurrent sequence (with $n = 0, 1, 2, 3, \ldots$),

$$\{F_{n+2} = F_n + F_{n+1}\}: 0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots. \tag{A.4}$$

These numbers have long since been known in biological phyllotaxis laws, whereby in numerous cases of symmetrical shaping of biological objects the numerical characteristics of their configurations are pairs of Fibonacci numbers from sequences of two types (parastichy and orthostichy):

$$\left\{Q_n' = \frac{F_{n+1}}{F_n}\right\}: \frac{2}{1}, \frac{3}{1}, \frac{5}{3}, \frac{8}{5}, \ldots \to \Phi = \frac{1+\sqrt{5}}{2} = 1.618\ldots \tag{A.5}$$

and

$$\left\{Q_n'' = \frac{F_{n+2}}{F_n}\right\}: \frac{2}{1}, \frac{3}{1}, \frac{5}{2}, \frac{8}{3}, \frac{13}{5}, \frac{21}{8}, \ldots \to \Phi^2 = \frac{3+\sqrt{5}}{2} = 2.618\ldots. \tag{A.6}$$

Various researchers have noted such regularities in the symmetry of numerous vegetable and animal objects (for a survey see Ref. [35, pp. 8–19]). Phyllotaxis has been discussed in a range of recent papers. A pioneering effort was made by Kepler who, in the seventeenth century, related in the laws of symmetrical morphogenesis with Fibonacci numbers and the golden section $\Phi = (1 + \sqrt{5})/2$ [see the sequence (A.5)], which has been known since ancient times and is so important in the esthetics of proportions.

Unlike sequences (A.5) and (A.6), the writer has studied elsewhere a sequence of cross-ratios [wurft, equation (3)] rather than affine or common ratios of neighboring number in the Fibonacci series. Three such neighboring numbers F_n, F_{n+1} and F_{n+2}, may be interpreted as lengths of three sequential stretches contained, in compliance with the recurrent property of the Fibonacci series (A.4), between four points F_{n+1}, F_{n+2}, F_{n+3} and F_{n+4}. The value of the wurf, equation (3), of such stretches is

$$W_n = \frac{(F_n + F_{n+1})\cdot(F_{n+1} + F_{n+2})}{F_{n+1}\cdot(F_n + F_{n+1} + F_{n+2})}. \tag{A.7}$$

Consequently, the wurf series has the form

$$\{W_n\}: 1, \frac{3}{2}, \frac{5}{4}, \frac{8}{6}, \frac{13}{10}, \ldots, \frac{F_{n+2}}{2F_n}, \ldots \to P = \lim_{n\to\infty} \frac{F_{n+2}}{2F_n} = \frac{\Phi^2}{2} = \frac{3+\sqrt{5}}{4} = 1.309\ldots. \tag{A.8}$$

In the same way that the limit of sequences of common ratios (A.5) of Fibonacci numbers is referred to as the golden section, the limit value P of the wurf sequence (A.8) obtained in this paper is referred to as the golden wurf. There are numerous convincing arguments in favor of the golden wurf $P \approx 1.31$ being the reference value of ontogenetically invariant wurf values of the three-membered kinematic blocks which comprise the human body (Section 5).

To conclude this appendix, iterative algorithms in a more or less explicit form are widely used in numerous applied mathematical fields, such as the theories of automata, iterations, optimal algorithms, multistep processes without "after-effects" in the theory of dynamic systems, Bellman's theory of dynamic programming, the ergodic theory, the theory of branching of solutions to nonlinear equations in cases of group invariance etc. Each of them has its own potential for interpreting the origin and biological significance of cyclomeric biostructures and for mathematization of biology.